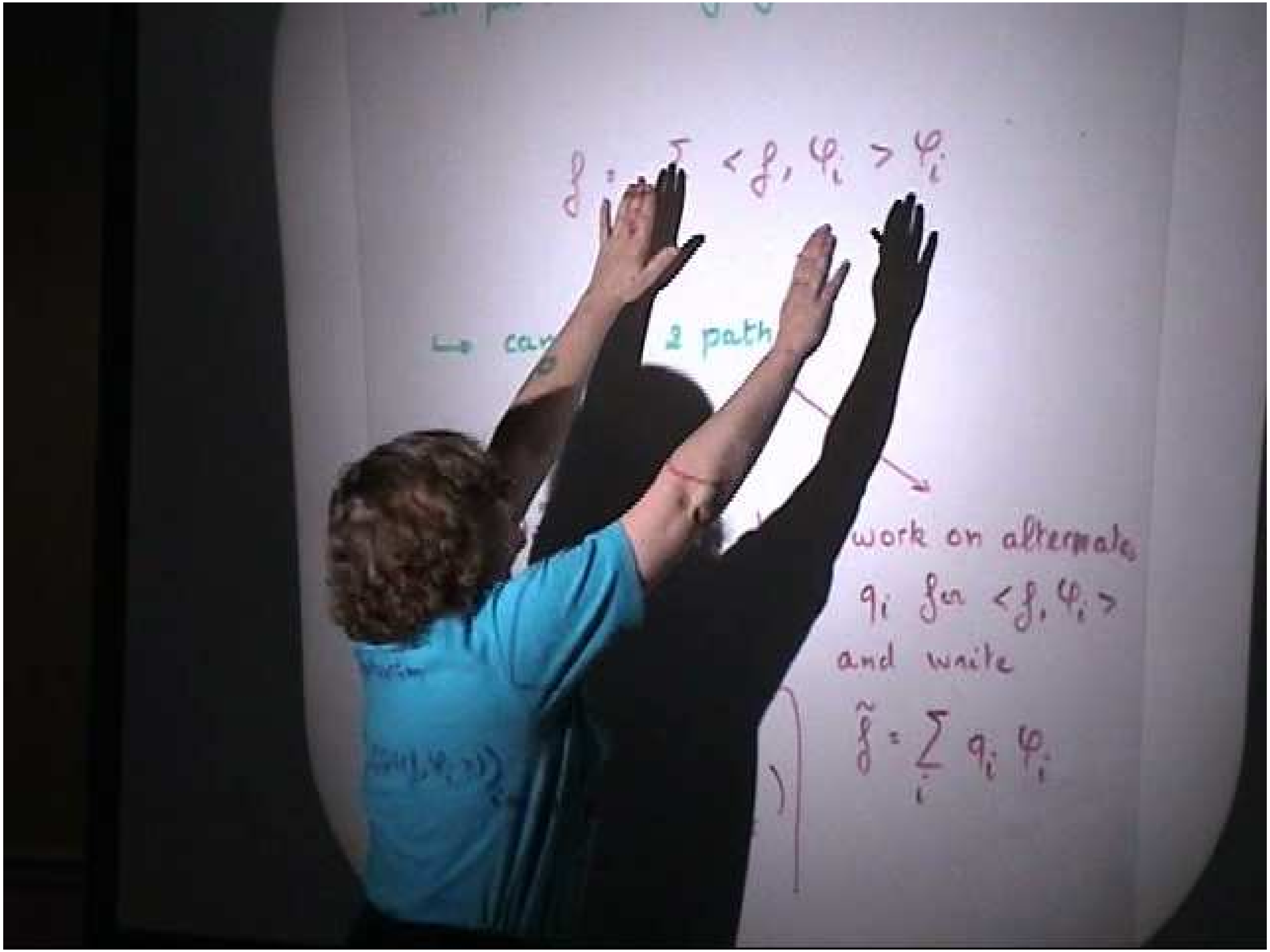
Greedy Algorithms for Manifold Recovery

Ronald DeVore

Texas A& M University

Some Personal Reflections

Ingrid Paying Homage to Her Work



Ingrid Under Arrest

Ingrid Celebrating Her Release

Ingrid Still Celebrating

What a Party Animal

Ingrid Auditioning

Ingrid Belting out Another Tune

Ingrid Getting Warmed Up

Ingrid Stealing the Show

Ingrid Grand Finale

Ingrid Biorthogonal

We got it Albert!

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  - Typically greedy algorithms do not perform well
  - We shall try to understand if this is the case for this algorithm

Reduced Basis

- Example for parametric elliptic problems

$$-\mathrm{div}(a\nabla_x u_a) = f(x), \quad x \in D, a \in \mathcal{A}$$
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  - Given a query $a \in \mathcal{A}$, fast on line solution by the Galerkin projection $P_n u_a$ onto X_n

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- Of course for the given tolerance ϵ we want N to be as small as possible

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- But before going into an analysis of this algorithm let us point out that as it stands, the algorithm is too idealized for applications and therefore we consider the following weak version of this algorithm

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- The discussion and analysis below applies to any realization of this algorithm

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- So Greedy Snapshots give same performance

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with a fixed C, c' and $\beta := \frac{\alpha}{\alpha+1}$

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- For the weak greedy algorithm with constant γ in a Banach pace X, we have the following:
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  - If $d_n(\mathcal{F}) \le C_0 n^{-\alpha}, \quad n = 1, 2, \ldots$, then for all $\beta > 1/2$

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- Think Kadec-Snowbar