Robustly Stable Signal Recovery in Compressed Sensing with Structure Matrix Perturbation

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Stable Signal Recovery of Standard CS

The standard Compressive Sensing (CS) problem is to recover signal $\mathbf{x}^o$ from compressed measurement

$$\mathbf{y} = \mathbf{\Phi}\mathbf{x}^o + \mathbf{e}$$

where $\mathbf{\Phi} \in \mathbb{R}^{m\times n}(m \ll n)$ is known and $\|\mathbf{e}\|_2 < \epsilon$.

Theorem (E. Candes, 2008*)

Define the k-restricted isometry constant (RIC) $\delta_k(\mathbf{\Phi})$ to be the smallest number satisfying

$$(1-\delta_k(\mathbf{\Phi}))\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Phi}\mathbf{v}\|_2^2 \leq (1+\delta_k(\mathbf{\Phi}))\|\mathbf{v}\|_2^2$$

for all k-sparse vectors $\mathbf{v}$. *Assume that $\delta_{2k}(\mathbf{\Phi}) < \sqrt{2}-1$, then an optimal solution $\mathbf{x}^*$ to the Basis Pursuit DeNoising (BPDN) problem*

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1, \quad s.t.\ \|\mathbf{y}-\mathbf{\Phi}\mathbf{x}\|_2 \leq \epsilon$$

satisfies $\|\mathbf{x}^*-\mathbf{x}^o\|_2 \leq C_0^{std}k^{-1/2}\|\mathbf{x}^o-\mathbf{x}^k\|_1 + C_1^{std}\epsilon$

where $\mathbf{x}^k$ is the k-sparse truncation of $\mathbf{x}^o$, and C_0^{std}, C_1^{std} are functions of $\delta_{2k}(\mathbf{\Phi})$.

*E. Candes, The Restricted Isometry Property and Its Implications for Compressed Sensing, Comptes Rendus Mathematique, 2008.

Robust Signal Recovery in Perturbed CS

In perturbed CS, $\mathbf{\Phi} = \mathbf{A} + \mathbf{E}$ where $\mathbf{A}$ is the known nominal sensing matrix and $\mathbf{E}$ is the unknown perturbation. Hence

$$\mathbf{y} = \mathbf{\Phi}\mathbf{x}^o + \mathbf{e} = \mathbf{A}\mathbf{x}^o + (\mathbf{E}\mathbf{x}^o + \mathbf{e}).$$

Theorem (M. Herman and T. Strohmer, 2010*)

Denote $\|\mathbf{E}\|_2^{(k)}$ the largest spectral norm taken over all k-column submatrices of $\mathbf{E}$, and similarly define $\|\mathbf{\Phi}\|_2^{(k)}$. Assume that there exist constants $\varepsilon_{\mathbf{E},\mathbf{\Phi}}^{(k)}, \epsilon$ and $\epsilon_{\mathbf{E},\mathbf{x}^o}$ such that $\frac{\|\mathbf{E}\|_2^{(k)}}{\|\mathbf{\Phi}\|_2^{(k)}} \leq \varepsilon_{\mathbf{E},\mathbf{\Phi}}^{(k)}$, $\|\mathbf{E}\mathbf{x}^o\|_2 \leq \epsilon_{\mathbf{E},\mathbf{x}^o}$ and $\|\mathbf{e}\|_2 \leq \epsilon$. Assume that $\delta_{2k}(\mathbf{\Phi}) < \frac{\sqrt{2}}{(1+\varepsilon_{\mathbf{E},\mathbf{\Phi}}^{(2k)})^2} - 1$ and $\|\mathbf{x}^o\|_0 \leq k$, then an optimal solution $\mathbf{x}^*$ to the Nominal BPDN (N-BPDN) problem

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1, \quad s.t. \ \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \epsilon + \epsilon_{\mathbf{E},\mathbf{x}^o}$$

achieves robust signal recovery with $\|\mathbf{x}^* - \mathbf{x}^o\|_2 \leq C^{ptb}(\epsilon + \epsilon_{\mathbf{E},\mathbf{x}^o})$ where C^{ptb} is a function of $\delta_{2k}(\mathbf{\Phi})$ and $\varepsilon_{\mathbf{E},\mathbf{\Phi}}^{(2k)}$.

*M. Herman and T. Strohmer, General Deviants: An Analysis of Perturbations in Compressed Sensing, IEEE J. Selected Topics in Signal Processing, 2010.

Structured Perturbation

In structured perturbation, we further assume that $\mathbf{E} = \mathbf{B}\mathrm{diag}(\boldsymbol{\beta}^o)$, where $\mathbf{B}$ is known a priori and $\boldsymbol{\beta}^o \in [-r, r]^n$. Hence

$$\mathbf{y} = \mathbf{\Phi}\mathbf{x}^o + \mathbf{e} = (\mathbf{A} + \mathbf{B}\mathrm{diag}(\boldsymbol{\beta}^o))\mathbf{x}^o + \mathbf{e} = [\mathbf{A}, \mathbf{B}]\begin{pmatrix}\mathbf{x}^o \\ \boldsymbol{\beta}^o \odot \mathbf{x}^o\end{pmatrix} + \mathbf{e} \triangleq \mathbf{\Psi}\mathbf{z}^o + \mathbf{e}.$$

In this way we transformed the perturbation into a signal of interest $\mathbf{z}^o$. Hence $\mathbf{z}^o$ can be stably recovered via Theorem 1:

$$\min_{\mathbf{z}} \|\mathbf{z}\|_1, \quad \text{s.t. } \|\mathbf{y} - \mathbf{\Psi}\mathbf{z}\|_2 \leq \epsilon$$

provided that $\delta_{4k}(\mathbf{\Psi}) < \sqrt{2} - 1$. We call this solver TPS-BPDN.

$\mathbf{z}^o = \begin{pmatrix}\mathbf{x}^o \\ \boldsymbol{\beta}^o \odot \mathbf{x}^o\end{pmatrix}$ is 2k-D-sparse, each part being k-sparse and sharing the same support. The disadvantage of TPS-BPDN is that the duplicated sparsity structure in $\mathbf{z}^o$ is not exploited.

Signal Recovery Under Structured Perturbation

Theorem

Define $\bar{\delta}_{2k}(\mathbf{\Psi})$ to be 2k-D-RIC, the smallest number satisfying
$(1-\bar{\delta}_{2k}(\mathbf{\Psi}))\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Psi}\mathbf{v}\|_2^2 \leq (1+\bar{\delta}_{2k}(\mathbf{\Psi}))\|\mathbf{v}\|_2^2$
for all 2k-D-sparse vectors $\mathbf{v}$. *Assume* $\bar{\delta}_{4k}(\mathbf{\Psi}) < (\sqrt{2(1+r^2)}+1)^{-1}$ *and* $\|\mathbf{e}\| \leq \epsilon$, *then an optimal solution* $(\mathbf{x}^*, \boldsymbol{\beta}^*)$ *to the Perturbed BPDN (P-BPDN) problem*

$$\min_{\mathbf{x}\in\mathbb{R}^n, \boldsymbol{\beta}\in[-r,r]^n} \|\mathbf{x}\|_1, \quad s.t.\ \|\mathbf{y}-(\mathbf{A}+\mathbf{B}\,diag(\boldsymbol{\beta}))\mathbf{x}\|_2 \leq \epsilon$$

satisfies that $\|\mathbf{x}^*-\mathbf{x}^o\|_2 \leq (C_0 k^{-1/2}+C_1)\|\mathbf{x}^o-\mathbf{x}^k\|_1 + C_2\epsilon$ *and*
$\|(\boldsymbol{\beta}^*-\boldsymbol{\beta}^o)\odot\mathbf{x}^k\|_2 \leq (D_0 k^{-1/2}+D_1)\|\mathbf{x}^o-\mathbf{x}^k\|_1 + D_2\epsilon$
where $C_0, C_1, C_2, D_0, D_1, D_2$ *are functions of* $\bar{\delta}_{4k}(\mathbf{\Psi})$ *and* r.

The above P-BPDN problem is nonconvex and not easily solved.
A simple method is an Alternating Algorithm AA-P-BPDN:

$$\mathbf{x}^{(j+1)} = \arg\min_{\mathbf{x}} \|\mathbf{x}\|_1, \quad \text{s.t. } \|\mathbf{y}-(\mathbf{A}+\mathbf{B}\text{diag}(\boldsymbol{\beta}^{(j)}))\mathbf{x}\|_2 \leq \epsilon$$

$$\boldsymbol{\beta}^{(j+1)} = \arg\min_{\boldsymbol{\beta}\in[-r,r]^n} \|\mathbf{y}-(\mathbf{A}+\mathbf{B}\text{diag}(\boldsymbol{\beta}))\mathbf{x}^{(j+1)}\|_2$$

Experiment

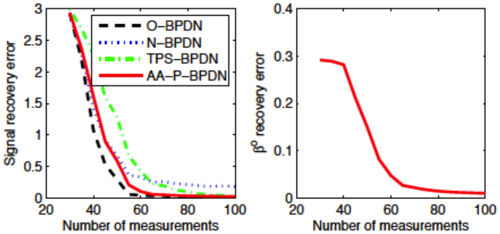

Fig. 4. Signal and perturbation recovery errors with respect to the number of measurements with parameter settings $(n, k, r, \epsilon) = (200, 10, 0.1, 0.2)$. AA-P-BPDN for P-BPDN in SP-CS has the best performance except the ideal case of O-BPDN.

Oracle BPDN (O-BPDN):

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1, \quad \text{s.t. } \|\mathbf{y} - (\mathbf{A} + \mathbf{B}\text{diag}(\beta^o))\mathbf{x}\|_2 \leq \epsilon$$

assuming β^o is known.