An Analysis of New and Existing FDTD Methods for Isotropic Cold Plasma and a Method for Improving Their Accuracy

Steven A. Cummer, *Student Member, IEEE*

***Abstract*— Over the past few years, a number of different finite-difference time-domain (FDTD) methods for modeling electromagnetic propagation in an isotropic cold plasma have been published. We have analyzed the accuracy and stability of these methods to determine which method provides the greatest accuracy for a given computation time. For completeness, two new FDTD methods for cold plasma, one of which is based on the concept of exponential fitting, are introduced and evaluated along with the existing methods. We also introduce the concept of cutoff modification which can be easily applied to most of the FDTD methods, and which we show can improve the accuracy of these methods with no additional computational cost. Von Neumann's stability analysis is used to evaluate the stability of the various methods, and their accuracy is determined from a straightforward time-and-space harmonic analysis of the dispersion and dissipation errors. Results of numerical experiments to verify the accuracy analysis are presented. It is found that for low-loss plasma, the PLRC method [4] is the most accurate, but the method of Young [1] can use less memory and is nearly as accurate. In this low-loss plasma regime, cutoff modification can significantly reduce the error near cutoff at the expense of slightly greater error at lower frequencies. For strongly collisional plasmas, the PLRC method also provides the most accurate solution.**

***Index Terms*—FDTD methods, plasmas.**

I. Introduction

Over the past few years, a number of different finite-difference time-domain (FDTD) methods to model electromagnetic propagation in isotropic cold plasma have been published [1]–[3], as have other FDTD methods for dispersive media that can be applied to an isotropic cold plasma [4]–[6]. Some of these techniques are based on direct finite-difference approximations of the complete field equations of the medium [1], [2] that consist of Maxwell's equations coupled to an auxiliary ordinary differential equation, which models the response of the current to the fields. These methods are commonly referred to as direct integration (DI) methods. The majority of the other methods are based on a difference approximation of Maxwell's equations coupled to an iteration derived from the convolution integral form of the auxiliary differential equation [3], [4]. This technique, called recursive convolution (RC), has also been applied to dispersive dielectrics [7], [8] and magnetized cold plasma [9]. Other methods are based on z-transforms [5] and the transmission-line matrix method [6].

The aim of this study is to determine, quantitatively, which of these methods is the most accurate for a fixed computational effort. To be all inclusive, we also develop two new FDTD methods for propagation in a cold plasma. We also suggest an improvement, which we call cutoff modification, that can be applied to some of the new and existing methods and can improve the accuracy significantly in certain parameter regimes. Briefly, it is a technique by which the medium parameters in the numerical simulation are slightly perturbed in such a way that the properties of the physical medium are better modeled by the numerical method.

As the two key qualities of an FDTD method are its stability and its accuracy, we evaluate both of these parameters for all of the new and existing methods considered herein. For the accuracy analysis, we derive the numerical dispersion relations for the different methods and compare them to the corresponding analytical relation, which yields the dispersion and dissipation error for each method. This method has been used with success in [12] and [13] for evaluating the accuracy of methods for dispersive dielectric. However, differing from these analyses, we attempt to evaluate the accuracy not for fixed Δt or Δz, but for a fixed computational effort. The operation count for each method considered is different allowing for a greater spatial and temporal resolution (leading to better accuracy) in some methods for fixed simulation run times. Since the ultimate goal of all numerical simulations is to produce the best answer in the least amount of time, this difference should be accounted for in the accuracy analysis.

II. Existing Methods

There are many different numerical methods for the simulation of electromagnetic wave propagation in a plasma. One method for the nonlinear propagation of high-power waves couples Maxwell's equations to the Boltzmann equation for the electron-velocity distribution function [14]. Other methods include fully kinetic particle simulations [15], magneto hydrodynamics [16], and hybrid particle-fluid methods [16]. Each of these techniques is appropriate under different physical conditions. In cold plasmas such as the ionosphere and magnetosphere, the description of low-power wave propagation is well approximated by magneto-ionic theory [17], which

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The author is with the Space, Telecommunications, and Radioscience Laboratory, Stanford University, Stanford, CA 94305 USA.

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assumes zero thermal velocity for the charged species. This is a good approximation as long as the thermal velocity is much less than the phase velocity of waves in the medium [18].

In magneto-ionic theory, the governing equations consist of Maxwell's equations coupled to an auxiliary equation relating current and electric field. This auxiliary equation is derived from the equation of motion of the charged particles in the wave electric field and (sometimes) an ambient constant magnetic field. The motion of ions can be neglected under many circumstances due to the ions' larger mass, and we consider only nonmagnetized (and, therefore, isotropic) electron plasmas in this paper. For a nonmagnetized cold plasma, the complete field equations are [17]

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{1}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \tag{2}$$

$$\frac{\partial \mathbf{J}}{\partial t} + \nu \mathbf{J} = \epsilon_0 \omega_P^2 \mathbf{E} \tag{3}$$

where ω_P is the plasma frequency and ν is the electron-neutral collision frequency. A completely equivalent form is used in [2], where the substitution $\mathbf{J} = \partial \mathbf{P}/\partial t$ is made.

As mentioned above, the two existing classes of FDTD methods are DI and RC. The leapfrog approximations to (1) and (2) are well known [19] and common to all of the DI methods. Young's DI method couples these leapfrog approximations, including the term $\mathbf{J}^{n+1/2}$ for $\mathbf{J}(t)$ in the difference equation for (2), to the difference equation

$$\frac{\mathbf{J}^{n+1/2} - \mathbf{J}^{n-1/2}}{\Delta t} + \nu \frac{\mathbf{J}^{n+1/2} + \mathbf{J}^{n-1/2}}{2} = \epsilon_0 \omega_P^2 \mathbf{E}^n \tag{4}$$

which is a second-order accurate approximation of (3). This method requires the storage of only one time level of each field component, and uses significantly fewer multiply and add operations per time step than the other methods described herein.

Nickisch and Franke's DI (NFDI) method [2] is slightly different. It includes the term $(\mathbf{J}^{n+1} + \mathbf{J}^n)/2$ for $\mathbf{J}(t)$ in (2) and approximates (3) by

$$\frac{\mathbf{J}^{n+1} - \mathbf{J}^{n-1}}{2\Delta t} + \nu \mathbf{J}^n = \epsilon_0 \omega_P^2 \mathbf{E}^n \tag{5}$$

which is also second-order accurate. However, the memory requirement for this method is greater than for Young's DI method, as two time levels of $\mathbf{J}$ must be stored.

The derivation of the recursive convolution methods is quite different, but the resulting iteration is a coupled set of three first-order difference equations, just as in the DI case. These methods are based on a time-domain integral relating $\mathbf{D}$ and $\mathbf{E}$ rather than an ordinary differential equation relating $\mathbf{J}$ and $\mathbf{E}$. For an isotropic cold plasma, the RC method is derived in [3], and we refer the reader there for details. An improved RC method, the piecewise linear recursive convolution method (PLRC), has been developed and applied to dispersive dielectrics [4]. This PLRC method can be straightforwardly extended to a cold isotropic plasma by using the time-domain susceptibility function from [3] to compute the necessary difference-equation coefficients from the formulas presented in [4]. Another RC method, the trapezoidal recursive convolution (TRC) method [7] has been developed and applied to dispersive dielectrics. However, this method is not directly applicable to isotropic cold plasmas [20] and thus is not considered in this paper.

It has been shown [4], [7] that for a second-order dispersive (Lorentz) dielectric, PLRC and TRC methods are much more accurate than the original RC method in [8]. Similarly, numerical tests show that the original RC method for cold plasmas is far less accurate than all of the other methods mentioned herein, so we do not consider this method in this analysis.

The z-transform method described in [5] requires five state variables for the iteration, and the TLM method in [6], when reduced to the isotropic plasma case, uses six state variables. As a result, these methods are more computationally intensive and more difficult to analyze than the DI and RC methods, which need only three state variables. For this reason, we do not analyze them here.

III. Two New Methods

A. A New DI Method

We present a new DI method for isotropic plasma to include in the comparisons that is very similar to the FDTD method for first-order dispersive dielectrics derived in [10]. This method, which we refer to as the new DI method, includes the term $(\mathbf{J}^{n+1} + \mathbf{J}^n)/2$ for $\mathbf{J}(t)$ in (2) and approximates (3) by

$$\frac{\mathbf{J}^{n+1} - \mathbf{J}^n}{\Delta t} + \nu \frac{\mathbf{J}^{n+1} + \mathbf{J}^n}{2} = \epsilon_0 \omega_P^2 \frac{\mathbf{E}^{n+1} + \mathbf{E}^n}{2}. \tag{6}$$

As with the other DI methods, it is second-order accurate.

B. Exponential Fitting

The method of exponential fitting was originally developed in [11] and was designed to provide accurate integration of ordinary differential equations (ODE's) with solution components that vary rapidly on the order of a time step. As an example, consider the ODE for current in (3) and its difference approximation in (6). It is easy to show that the homogeneous solution of this difference equation has a growth per time-step factor of $(1 - \nu\Delta t/2)/(1 + \nu\Delta t/2)$. Notice, however, that the homogeneous solution of the original differential equation varies as $e^{-\nu t}$ and, thus, has a growth per time-step factor of $e^{-\nu\Delta t}$. It is immediately apparent from a comparison of these eigenvalues that (6), when treated as a single, uncoupled difference equation, is a good approximation of (3) only for $\nu\Delta t \ll 1$.

In contrast, the application of one-step exponential fitting to (3) yields the difference equation

$$\begin{aligned}\mathbf{J}^{n+1} = {} & e^{-\nu\Delta t}\mathbf{J}^n + \frac{\epsilon_0 \omega_P^2}{\nu^2 \Delta t}\left[\left(\nu\Delta t + e^{-\nu\Delta t} - 1\right)\mathbf{E}^{n+1}\right. \\ & \left. + \left(1 - e^{-\nu\Delta t} - \nu\Delta t e^{-\nu\Delta t}\right)\mathbf{E}^n\right]\end{aligned} \tag{7}$$

the homogeneous solution of which has a growth per time-step factor of $e^{-\nu\Delta t}$, and is, therefore, exponentially fitted to the original ODE. Thus, treated as uncoupled difference equations,

(7) provides a more accurate integration of (3) than (6) for nonsmall $\nu\Delta t$. From this, we might expect the exponentially fitted equation (7) to give a more accurate FDTD method for nonsmall $\nu\Delta t$ when coupled to difference approximations of (1) and (2) than the DI methods. However, as the accuracy analysis in Section V will show, the exponentially fitted and the various DI methods are comparably accurate for the entire range of plasma parameters. Like the new DI method, this method requires two levels of $\mathbf{J}$ storage.

As a notable aside, it can be shown that the RC method in [3] is exactly equivalent to a particular first-order accurate discretization using exponential fitting. This equivalence demonstrates directly that this RC method is only first-order accurate, a fact which is suggested by certain assumptions in the original derivation of the methods and was pointed out in [12]. Also, it was suggested in [3] that the recursive convolution method for plasma would not be accurate at zero frequency due to a modification to the frequency-domain permittivity that was made during the derivation. The equivalent derivation using exponential fitting makes no such restriction, demonstrating that the method is indeed applicable to fields with a dc component.

IV. Stability Analysis

One of the two key properties that determines the utility of any difference approximation is stability. If a method is unstable, then some (or all) spatial frequency components of the solution grow with time and eventually the desired solution. The simplest stability analysis technique is von Neumann stability analysis [20]. Strictly speaking, this analysis is only applicable to problems with periodic boundary conditions, and far more complicated techniques exist for evaluating stability for a true initial-boundary value problem (e.g., [21]). However, we are concerned solely with the stability of the methods independent of boundary conditions, so von Neumann analysis suits our needs.

We consider fields varying in one dimension for this stability analysis, propagating in the $\hat{z}$ direction with nonzero components H_y, E_x, and J_x. The spatially harmonic form of the difference equations (found by assuming an $e^{ik_{\text{num}}m\Delta z}$ variation in space) can be written in the general form $\mathbf{A}v^{n+1} = \mathbf{B}v^n$, where $\mathbf{A}$ and $\mathbf{B}$ are matrices and v is a column vector containing all of the field components. The field growth per time-step factors are then given by the eigenvalues of the matrix $\mathbf{A}^{-1}\mathbf{B}$. If any of these eigenvalues are larger than unity for any $k \leq 2\pi/\Delta z$, then the method is unstable. Alternatively, one could find the characteristic polynomial of $\mathbf{A}^{-1}\mathbf{B}$ and determine these eigenvalues analytically as the zeros of this polynomial (an analysis of this type was performed in [13] for FDTD methods for dispersive dielectrics). However, we found the numerical method to be much simpler to implement and equally accurate, so we have used it here.

We have numerically determined the maximum stable Courant number (defined as $c_0\Delta t/\Delta z$) as a function of the plasma medium parameters $\omega_P\Delta t$ and $\nu\Delta t$ using this procedure. For the new DI and EF methods, the maximum stable Courant number is unity for all $\omega_P\Delta t$ and $\nu\Delta t$, just as if the medium were free space. For Young's DI method, the stability requirement is slightly more restrictive. In [1], by using a method outlined in [12], it was shown that for zero collision frequency, the maximum stable Courant number is given by $\sqrt{1-(\omega_P\Delta t/2)^2}$. We find from the numerical stability analysis that this result is valid for all nonzero $\nu\Delta t$ as well.

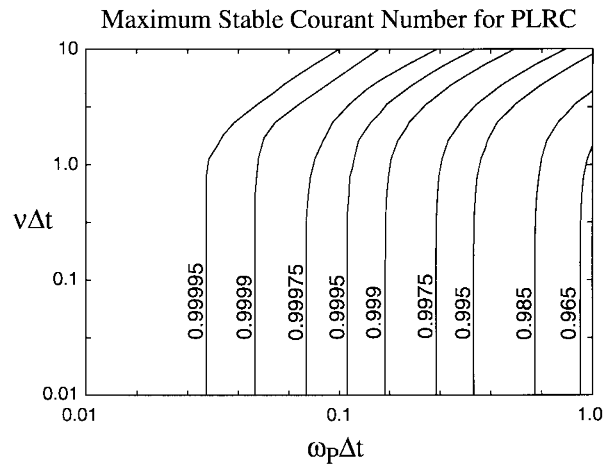

Fig. 1. A contour plot of the maximum stable courant number for the PLRC method as a function of $\nu\Delta t$ and $\omega_P\Delta t$.

For the NFDI method with zero collision frequency, the maximum stable Courant number is the same as for Young's DI method, $\sqrt{1-(\omega_P\Delta t/2)^2}$. However, if $\nu\Delta t \neq 0$, then the method is unconditionally unstable for all Courant numbers. This result is not that surprising, as it is easy to show that the difference approximation for the current equation in this method (5), when treated as a stand-alone ordinary difference equation, is unstable for all nonzero real ν [22]. The instability becomes more severe for increasing $\nu\Delta t$.

The PLRC method also has a stability criterion that varies with the medium parameters but that cannot be simply summarized. Fig. 1 shows a contour plot of the maximum stable Courant number as a function of $\omega_P\Delta t$ and $\nu\Delta t$. The maximum stable Courant number is always less than unity and becomes significantly less than unity as $\omega_P\Delta t$ increases. We have not considered values of $\nu\Delta t > 10$, for at this point $\nu \gg \omega$ for all ω that can exist in the finite-difference system. In this regime, as can be seen from (3), the dJ/dt term becomes insignificant compared to νJ, and to a good approximation the medium becomes simply conducting with conductivity $\epsilon_0\omega_P^2/\nu$.

Interestingly, the maximum stable Courant number for the original RC method is actually greater than unity for all tested medium parameters. However, the relatively low accuracy of this method limits any practical utility of this property.

All of the stability results from this section are summarized in Table I.

V. Accuracy Analysis

Accuracy is the second key property of a finite-difference method. A straightforward technique that has been previously used for determining the accuracy of FDTD methods [12], [13]

TABLE I
NUMERICALLY DERIVED STABILITY LIMITS

Method	Max. Courant Number
Young's DI	$\sqrt{1-(\omega_P\Delta t/2)^2}$
PLRC	always < 1 (see Figure 1)
NFDI	if $\nu = 0$, $\sqrt{1-(\omega_P\Delta t/2)^2}$, else unstable
New DI	1
Exponential Fitting	1

involves the construction of the dispersion relation relating the index of refraction n (or equivalently, the wave number k) to frequency for waves in the finite-difference system and a subsequent comparison with the dispersion relation for the differential equations. This method allows easy evaluation of both the dispersion (real part of n) and dissipation (imaginary part of n) errors as a function of medium and numerical (e.g., points per wavelength) parameters. We do not include the NFDI method in this analysis, as it was shown to be unstable for $\nu \neq 0$, and for $\nu = 0$, its accuracy and stability properties are identical to Young's DI method.

Assuming $e^{i(\omega t - kz)}$ variation for all of the field quantities, the analytic plasma dispersion relation is, from (1)–(3)

$$c_0^2 k^2 = \omega^2 - \frac{\omega_P^2}{1 - i\,\dfrac{\nu}{\omega}}. \tag{8}$$

The dispersion relations satisfied by the finite-difference approximations are found by assuming $e^{i(\omega l\Delta t - k_{\mathrm{num}} m\Delta z)}$ variation for the field quantities, with l and m enumerating the time step and spatial grid point, respectively. Let $K = (c_0\Delta t/\Delta z)^2 \sin^2(k_{\mathrm{num}}\Delta z/2)$. For the new DI finite-difference method, the numerical dispersion relation is

$$K = \sin^2\left(\frac{\omega\Delta t}{2}\right) - \frac{\left(\dfrac{\omega_P\Delta t}{2}\right)^2 \cos^2\left(\dfrac{\omega\Delta t}{2}\right)}{1 - i\,\dfrac{\nu\Delta t}{2}\cot\left(\dfrac{\omega\Delta t}{2}\right)}. \tag{9}$$

The dispersion relation for Young's direct integration method is [1]

$$K = \sin^2\left(\frac{\omega\Delta t}{2}\right) - \frac{\left(\dfrac{\omega_P\Delta t}{2}\right)^2}{1 - i\,\dfrac{\nu\Delta t}{2}\cot\left(\dfrac{\omega\Delta t}{2}\right)}. \tag{10}$$

The dispersion relation for the exponentially fitted method is

$$K = \sin^2\left(\frac{\omega\Delta t}{2}\right) - \frac{\left(\dfrac{\omega_P\Delta t}{2}\right)^2}{\nu\Delta t}\, i\sin(\omega\Delta t) \times \left[1 - \frac{2\sinh\left(\dfrac{\nu\Delta t}{2}\right)\sin\left(\dfrac{\omega\Delta t}{2}\right)}{\nu\Delta t \sin\left(\dfrac{\omega\Delta t}{2} - i\,\dfrac{\nu\Delta t}{2}\right)}\right]. \tag{11}$$

Finally, the dispersion relation for the PLRC method is

$$K = (1-\xi_0)\sin^2\left(\frac{\omega\Delta t}{2}\right) + \frac{1}{4}\chi_0(1 - e^{i\omega\Delta t}) - \frac{\dfrac{1}{4}\Delta\chi_0(1 - e^{i\omega\Delta t}) - \Delta\xi_0 \sin^2\left(\dfrac{\omega\Delta t}{2}\right)}{e^{i\omega\Delta t} - e^{-\nu\Delta t}} \tag{12}$$

where the expressions for ξ_0, $\Delta\xi_0$, χ_0, and $\Delta\chi_0$ are given in [4].

From these relations, one can easily solve for the index of refraction $n(\omega) = ck(\omega)/\omega$ and then compare these numerical indexes to those of the analytical problem.

VI. Improved Accuracy via Cutoff Modification?

It was noted in [1] that the cutoff frequency (i.e., frequency for which the wave number k goes to zero and propagation stops) for the collisionless finite-difference field equations using Young's DI method is $\omega_{co} = (2/\Delta t)\sin^{-1}(\omega_P\Delta t/2)$, which is slightly different from that of the collisionless analytic field equations, $\omega_{co} = \omega_P$. This difference increases with increasing $\omega_P\Delta t$, which is a problem because one often wishes to run an actual simulation with the largest possible $\omega_P\Delta t$ to keep the simulation as fast as possible. It is a simple matter to change the plasma frequency used in the FDTD model artificially to move the numerical cutoff frequency to the physical cutoff frequency. Applying this principle to Young's DI method, we arrive at the relation $\omega_P^{\mathrm{num}} = (2/\Delta t)\sin(\omega_P\Delta t/2)$, where ω_P is the actual plasma frequency to be simulated and ω_P^{num} is the plasma frequency to be used in the numerical method. Similarly, for the new DI method for which the cutoff condition is $\omega = (2/\Delta t)\tan^{-1}(\omega_P\Delta t/2)$, this cutoff modification procedure yields $\omega_P^{\mathrm{num}} = (2/\Delta t)\tan(\omega_P\Delta t/2)$. The application of this cutoff modification to the PLRC method is more complicated due to the more complicated dispersion relation, and we do not investigate it here.

This modification is an attempt to try to make the simulation model the physics of the system more accurately, and we show below that it indeed improves the numerical solution under common conditions. Note that this change does not add any complexity or additional work to the algorithm; it only involves changing the plasma frequency slightly in the hopes of getting a more accurate simulation.

VII. Accuracy Analysis Results

We are now prepared to evaluate the accuracy of these different methods for different plasma parameter regimes by comparing the numerical and analytic dispersion relations. All of the numerical dispersion relations depend on the dimensionless plasma parameters $\omega_P\Delta t$ and $\nu\Delta t$ and the dimensionless simulation parameters $\omega\Delta t$, $k\Delta z$, and $c_0\Delta t/\Delta z$. Given these values, $n_{\mathrm{num}}(\omega)$ can be determined and compared with the analytical expression $n_{\mathrm{anal}}(\omega)$.

To help describe the errors in a given simulation, we examine dispersion and dissipation error as a function of frequency for fixed medium parameters. The difference equations for each of the four methods evaluated were rearranged so that

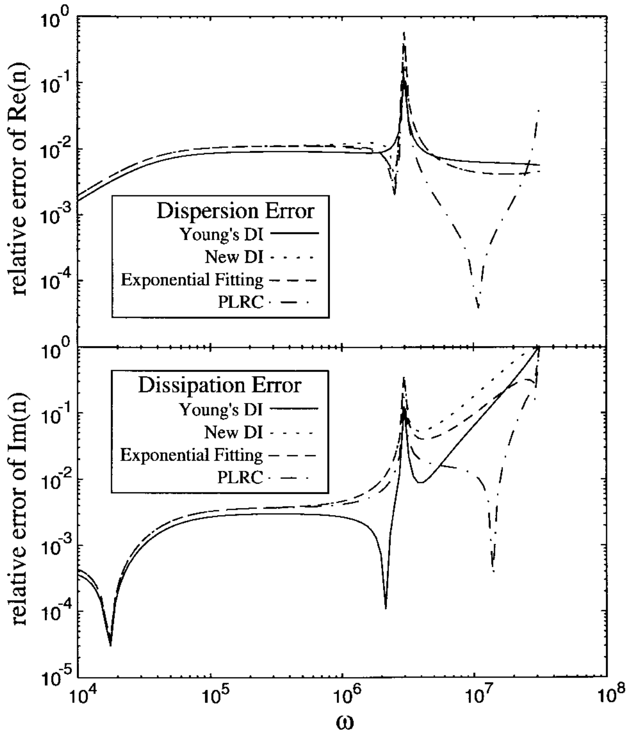

Fig. 2. Plots of the dispersion and dissipation errors as functions of ω for the $\nu\Delta t \ll 1$ case.

each method requires only one level of storage for each field variable, which slightly increases the multiply and add count (except for Young's DI method, which is already in this form), but saves copy operations. As advertised, we account for the differing computational requirements of each in this analysis by reducing Δt and Δz proportionally in the simulation using Young's DI (which is faster than the other three methods), thereby increasing accuracy but slowing execution. The time- and space-step values used for each method are as follows: the new DI and EF methods both use $\Delta t = 10^{-7}$ s and $\Delta z = 30$ m, the PLRC method uses $\Delta t = 0.9962 \times 10^{-7}$ s and $\Delta z = 30$ m, and Young's DI method uses $\Delta t = 0.900\,75 \times 10^{-7}$ s and $\Delta z = 27.2727$ m. These values were chosen so that a simulation of the same analytical problem with predetermined ν and ω_P will take approximately the same amount of time (within 8%) for each method and to provide the maximum stable Courant number for each method for $\omega_P = 3 \times 10^6$, which is used in all of the simulations and analyses.

We have divided this analysis into two regimes: $\nu\Delta t \ll 1$ and $\nu\Delta t \gtrsim 1$.

A. $\nu\Delta t \ll 1$

For the $\nu\Delta t \ll 1$ case, the plasma parameters are chosen to be $\nu = 3 \times 10^4$ and $\omega_P = 3 \times 10^6$. Using Δt and Δz from above gives $\omega_P \Delta t \approx 0.3$ and $\nu\Delta t \approx 0.003$ for all of the methods. This choice of $\omega_P\Delta t$ is close to as large as it can be and still yield accurate results (see Section VII-D below).

The total error can be separated into two parts: the dispersion error, which controls the phase error of each frequency component, is defined as $Re(n_{\rm num} - n_{\rm anal})/Re(n_{\rm anal})$, and the dissipation error, which controls the amplitude error of each frequency component, is defined as $Im(n_{\rm num} - n_{\rm anal})/Im(n_{\rm anal})$. Fig. 2 shows the dispersion error and the dissipation error as functions of ω for the above parameter values. For perspective, this range of $\omega = 10^4$–10^7 corresponds to a sampling of ≈6000–6 points-per-period. Below the plasma frequency, Young's DI method has slightly lower errors than the other three methods due to the ≈10% smaller Δt and Δz. Had we not accounted for the different computational requirements, the errors would have been even closer. Young's DI and the PLRC methods have lower peak errors at the plasma frequency than the others, and above the plasma frequency, the PLRC method has the lowest error.

Strictly on the basis of accuracy, and considering that most of the error in a real simulation would come from frequencies near and above the plasma frequency, the PLRC method is the best. However, accounting for memory requirements makes the choice less clear. A one-dimensional implementation of Young's DI method contains two coefficients that depend on the medium parameters ν and ω_P, while a equivalent implementation of the PLRC method requires five of these coefficients. In a simulation with an arbitrarily inhomogeneous plasma in which these coefficients are different from cell to cell, these coefficients should be computed ahead of time and stored along with the field values. Such a simulation with the PLRC method would thus require significantly more memory than one with Young's DI method. Clearly, which method is superior depends on the intended use.

B. $\nu\Delta t \gtrsim 1$

Next, we consider the plasma parameter regime where $\nu\Delta t \gtrsim 1$. The simulation parameters are the same as above, except that we increase ν by a factor of 1000 to 3×10^7, yielding $\nu\Delta t \approx 3.0$ for all of the methods. Fig. 3 shows the dispersion error and the dissipation error versus frequency for the various methods.

The four methods are fairly close in dispersion error. However, they differ more strongly in dissipation error, which is much larger than the dispersion error for much of the examined frequency range. Young's DI method, even accounting for the smaller grid spacing, is the least accurate of the three for this parameter regime, while the PLRC has the greatest accuracy. That the two DI methods do as well as the EF method is somewhat surprising in the light of the previous observation that for $\nu\Delta t \gtrsim 1$, the integration of the equation for current is highly inaccurate for the direct integration methods.

C. Analysis of Cutoff Modification

Notice that for $\nu\Delta t \ll 1$, there is a peak in both the dispersion and dissipation errors in all the methods near $\omega \approx \omega_P$. This characteristic arises from the cutoff of the medium not being accurately modeled. As discussed in Section VI, a small change can be made to the value of ω_P used in the simulation that will move the numerical cutoff frequency to the proper physical cutoff frequency. We now investigate the effect that this change has on the accuracy of Young's DI method and the new DI method. We use the same simulation parameters as above for the $\nu\Delta t \ll 1$ case.

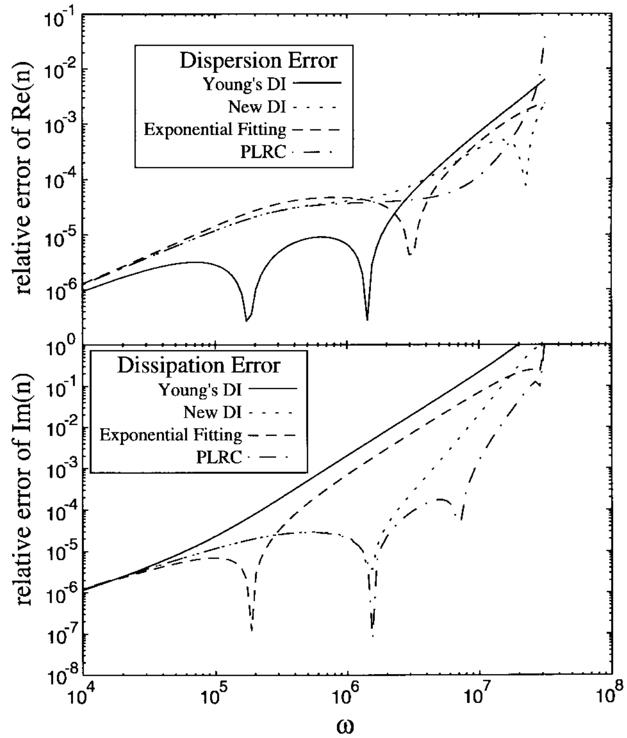

Fig. 3. Plots of dispersion and dissipation error as functions of ω for the $\nu\Delta t \gtrsim 1$ case.

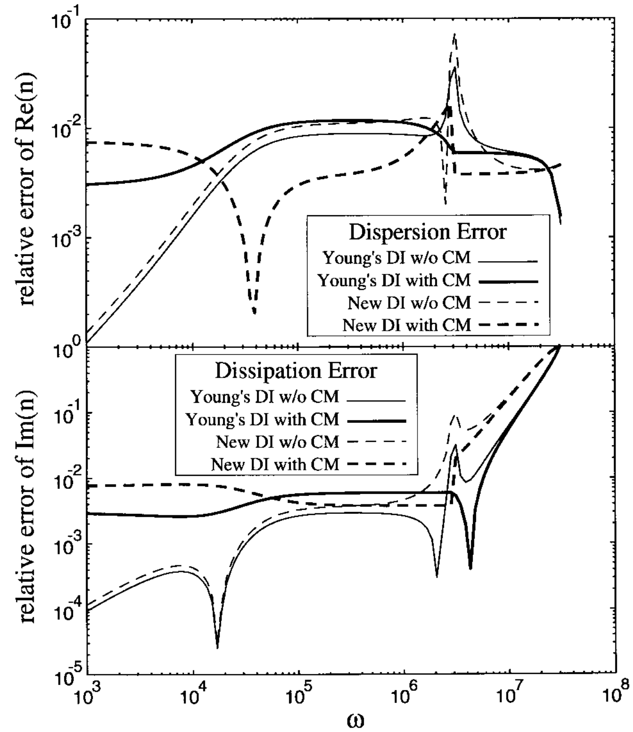

Fig. 4. A plot of dispersion and dissipation error versus ω for Young's DI and the new DI methods with and without cutoff modification. Cutoff modification reduces the upward spikes near the plasma frequency at the expense of reduced accuracy at lower frequencies.

Fig. 4 shows the dispersion and dissipation error with and without cutoff modification. For Young's DI method, the accuracy for $\omega < \omega_P$ for the modified version is worse, but the sharp upward spike near $\omega = \omega_P$ has been eliminated in both the dispersion and dissipation error. For the new DI method, the dispersion error for $\omega < \omega_P$ for the modified version is actually the same or better until $\omega \ll \omega_P$. The sharp upward spike near $\omega = \omega_P$ is also smoothed, though not quite to the extent as for Young's DI method.

There is a sharp change in the index of refraction in a plasma as long as $\nu \ll \omega_P$ (strictly speaking, a true cutoff exists only for $\nu = 0$, but there is still a "cutoff" where the wavelength becomes very large near the plasma frequency if $\nu \ll \omega_P$). This analysis shows that the cutoff modification does a good job of reducing the maximum dispersion and dissipation error at the expense of an accuracy reduction at low frequencies. Reduction of the dispersion error near the cutoff is especially important, as these frequencies are the most dispersed due to their slow group velocity and tend to be spread over the largest area of time and space in a simulation. For this reason, this accuracy tradeoff is often a good one to make. However, as $\nu \to \omega_P$, the benefit of cutoff modification is substantially reduced because the sharpness of the cutoff in the medium drops significantly and there is a smaller peak in the numerical error.

D. Accuracy versus $\omega_P\Delta t$

The accuracy of the various methods also clearly depends on the parameter $\omega_P\Delta t$, but the numerical dispersion relations for all of the methods depend similarly on this parameter. We have analyzed the dispersion and dissipation error as a function of $\omega_P\Delta t$ for a constant ω, and the results are straightforward. The dispersion error increases linearly with $\omega_P\Delta t$ (except for slightly more complicated behavior near $\omega_P = \omega$), and the dissipation error is nearly constant with respect to $\omega_P\Delta t$. This dependence can also be demonstrated by a Taylor series analysis of the numerical dispersion relations, but we do not present such here. Further analysis shows that this general dependence of the dispersion and dissipation error on the parameter $\omega_P\Delta t$ is independent of the value of $\nu\Delta t$. This result indicates that along with the usual sampling points per wavelength, $\omega_P\Delta t$ is a key parameter in determining the accuracy of a given simulation for all of the methods considered.

VIII. Numerical Experiments

We now present the results of numerical tests of the different methods to substantiate the error analysis of the previous section. The problem to be solved is the reflection and transmission at a smooth interface of free space and the medium in question. As our accuracy analysis above is based on errors as a function of temporal frequency, we will examine the fields as a function of time for a fixed point in the plasma medium as this allows an easier comparison between the accuracy analysis and numerical experiments.

To compare the numerical solutions to a reference solution, simulations were run using the new DI method and the time and space steps were successively decreased while maintaining $c_0\Delta t/\Delta z = 1$. The reference solutions were calculated using time and space steps 100 times smaller than those used for

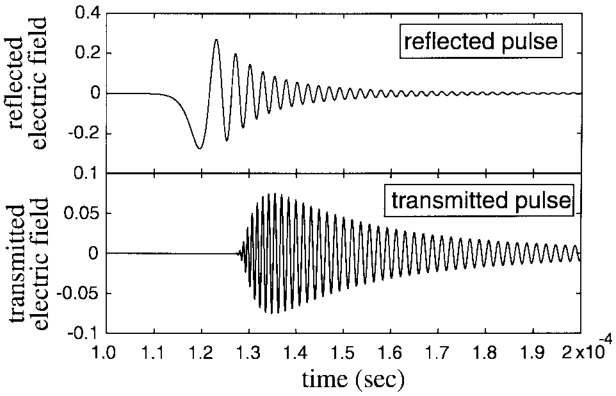

Fig. 5. Plots of the reference solution reflected and transmitted pulses for the $\nu\Delta t \ll 1$ problem.

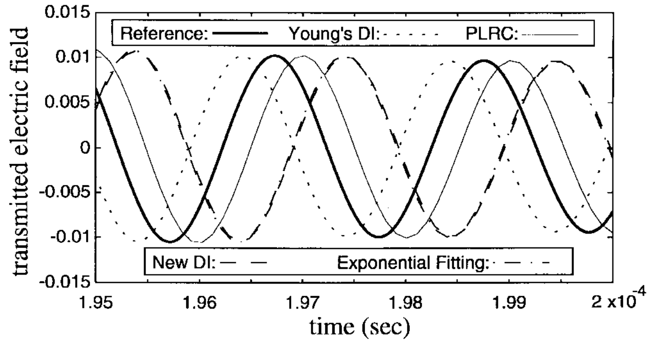

Fig. 6. A plot of the numerical solutions computed by the various methods of the late-time transmitted fields. The reference solution is shown for comparison. The EF and new DI methods are nearly identical.

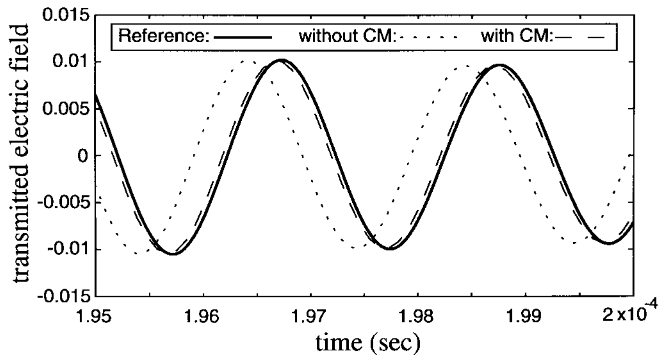

Fig. 7. A plot of the numerical solution computed using Young's DI method with and without cutoff modification. The reference solution is shown for comparison.

the test solutions. The stability and consistency of the new DI method guarantees convergence to the exact solution, and because of the second-order accuracy of this method, the reference solution has ≈10 000 times smaller error than the test solutions. Examination of the convergence of the solution as Δt and Δz were decreased showed this expected convergence rate, and the error in the reference solution is safely insignificant compared to that in the test solutions.

For each problem, a unity amplitude Gaussian pulse propagating in free space is incident on a boundary with increasing plasma and collision frequencies having a hyperbolic tangent dependence with distance, which creates a smooth step interface. The region of interest is 60 km long, with the plasma beginning at 27 km. The electric field is sampled each time step at 24 km and 39 km to examine the reflected and transmitted fields, respectively. The field variations with time at these output points are compared to the reference solutions. A Gaussian pulse was chosen to have a nonzero dc component to show that all of the plasma FDTD methods are valid for zero frequency. The pulse width is such that the spectral amplitude of the pulse is down by a factor of 100 (compared to the zero-frequency amplitude) at $\omega = 10^7$.

We again consider two plasma regimes: $\nu\Delta t \ll 1$ and $\nu\Delta t \gtrsim 1$.

A. $\nu\Delta t \ll 1$

For this case, we use $\omega_P^{\max} = 3 \times 10^6$ s^{-1}, $\nu^{\max} = 3 \times 10^4$ s^{-1}, and Δt and Δz, as in Section VII. Fig. 5 shows the reflected and transmitted pulses for the low-$\nu\Delta t$ reference solution. Note the expected qualitative characteristics for a low-loss plasma: low frequencies reflected, high frequencies transmitted, and the late-time transmitted and reflected fields approach the maximum plasma frequency, where the group velocity is slowest.

Fig. 6 is a close up of the late-time transmitted solutions for the various methods along with the reference solution. The general envelope of all of the solutions is fairly accurate, but there is significant phase (dispersion) error present in all of the numerical solutions. This dispersion error is hardly noticeable in the early part of the transmitted pulse but worsens with time. This is a consequence of the peak in the dispersion error near $\omega = \omega_P$ for all of the methods (see Fig. 2) which occurs in the frequencies with the lowest group velocity. Young's DI and the PRLC methods show the least error, primarily due to a lower peak error at $\omega = \omega_P$ for these methods compared to the new DI and the EF methods.

Examining the reflected pulses gives the same results. The methods all yield generally good agreement with the reference solution, though also with some significant phase error occurring in the later times similar to the error in the transmitted pulses due to peak error near the plasma frequency.

B. Cutoff Modification

Does the cutoff modification described in Section VI improve the solution? We find that the answer is a qualified affirmative.

Fig. 7 shows the late-time transmitted fields for the reference solution and as computed using Young's DI method with and without cutoff modification. The difference is quite clear: altering the plasma frequency so that the physical cutoff is better modeled by the numerical methods significantly improves the accuracy of the method near the plasma frequency. Examination of the early time transmitted pulse shows that cutoff modification even slightly improves the numerical solution for frequencies above the plasma frequency, as was demonstrated in Fig. 4. The reflected pulse yields similar results: cutoff modification improves the solution near the

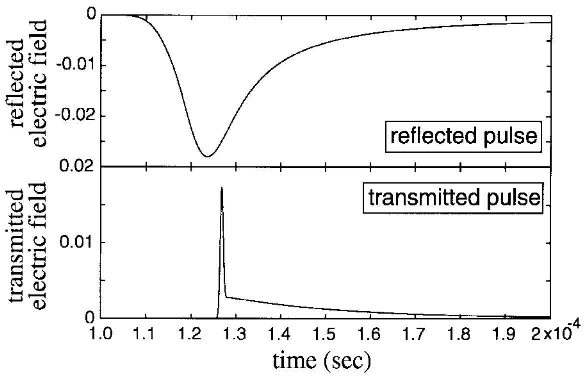

Fig. 8. Plots of the reference solution reflected and transmitted pulses for the $\nu\Delta t \gtrsim 1$ problem.

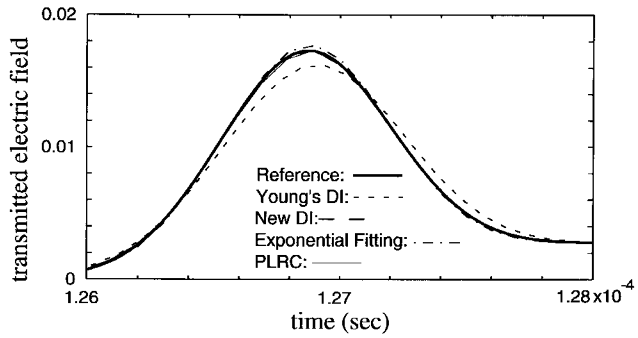

Fig. 9. A plot of the numerical solutions computed by the various methods of the early-time transmitted fields. The reference solution is shown for comparison. The new DI and PLRC solutions nearly coincide with the reference solution, and the PLRC solution is the most accurate.

plasma frequency, and does not affect it much for frequencies lower than ω_P.

This confirms the analysis in Section VII-C which showed that cutoff modification significantly reduces the maximum error (at $\omega \approx \omega_P$). This cutoff-modified simulation did not show the effect of the expected increase in error at low frequency, but it was not designed to do so. The application of cutoff modification to the new DI method yields similar results. Further analysis indicates that cutoff modification provides a significant benefit as long as $\nu/\omega_P \lesssim 0.1$.

C. $\nu\Delta t \gtrsim 1$

Fig. 8 shows the reflected and transmitted pulses for the high-$\nu\Delta t$ reference solution. The characteristics of the medium have changed drastically from the low-$\nu\Delta t$ case. High frequencies suffer little dispersion (the main transmitted pulse width is nearly the same as the incident pulse width) and significant loss, while low frequencies are strongly dispersed, as can be seen by the long tail of the transmitted pulse. These characteristics are clearly approaching those of a simple conductor.

As for the $\nu\Delta t \ll 1$ case, this simulation highlights the errors at the frequencies of maximum error, which from Fig. 3 can be seen to be the higher frequencies ($\omega > 10^6$). Fig. 9 is a close up of the early-time transmitted solutions for the various methods along with the reference solution. Young's DI method and the EF method solutions show primarily dissipation error (the pulses are minimally shifted or spread but show an amplitude error), and the new DI and PLRC methods have almost no error at all. This verifies the dissipation error results calculated in Section V and shown in Fig. 3—for $\nu\Delta t \gtrsim 1$, PLRC is the most accurate method, followed closely by the new DI method.

We do not examine the reflected pulse because it contains only low-frequency components for which the accuracy is very high.

IX. Summary and Conclusion

We have analyzed the stability and accuracy of three existing and two new (presented herein) FDTD methods for electromagnetic wave propagation in an isotropic cold plasma. The stability analysis showed that the method of Nickisch and Franke [2] is unstable for plasmas with nonzero collision frequency, which limits the usefulness of this method. The other methods considered had maximum Courant numbers close to or equal to unity.

In the accuracy analysis, we evaluated the accuracy of the various methods as a function of computational effort. This type of analysis is more practically useful than one using a fixed Δt and Δz, as it answers the question of which method provides the greatest accuracy in a fixed amount of time. Using a dispersion error analysis, we found that for low-loss ($\nu\Delta t \ll 1$) plasmas, the PLRC method [4] is the most accurate. However, Young's DI method nearly as accurate, is algorithmically simpler, and can be more memory efficient, so the selection of the more appropriate method would depend on the application. For lossy ($\nu\Delta t \gtrsim 1$) plasmas, the PLRC method was the most accurate, followed closely by the new DI method presented herein. These conclusions were verified by numerical experiments.

We also introduced cutoff modification, a technique for improving the accuracy of a low-loss simulation at no additional computational cost. It is an attempt to model the physical cutoff of the cold plasma medium better by perturbing slightly the value of the plasma frequency used in the simulation. Through both accuracy analysis and numerical experiments, it was shown that cutoff modification can yield a significant accuracy improvement for plasmas exhibiting sharp cutoff characteristics and containing excitation frequencies near the plasma frequency.

References

[1] J. L. Young, "A full finite difference time domain implementation for radio wave propagation in a plasma," *Radio Sci.*, vol. 29, pp. 1513–1522, 1994.
[2] L. J. Nickisch and P. M. Franke, "Finite-difference time-domain solution of Maxwell's equations for the dispersive ionosphere," *IEEE Antennas Propagat. Mag.*, vol. 34, pp. 33–39, Oct. 1992.
[3] R. J. Luebbers, F. Hunsberger, and K. S. Kunz, "A frequency-dependent finite-difference time-domain formulation for transient propagation in a plasma," *IEEE Trans. Antennas Propagat.*, vol. 39, pp. 29–34, Jan. 1991.
[4] D. F. Kelley and R. J. Luebbers, "Piecewise linear recursive convolution for dispersive media using FDTD," *IEEE Trans. Antennas Propagat.*, vol. 44, pp. 792–797, June 1996.
[5] D. M. Sullivan, "Z-transform theory and the FDTD method," *IEEE Trans. Antennas Propagat.*, vol. 44, pp. 28–34, Jan. 1996.

[6] T. Kashiwa, N. Yoshida, and I. Fukai, "Transient analysis of a magnetized plasma in three-dimensional space," *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 1096–1105, Aug. 1988.
[7] R. Siushansian and J. LoVetri, "A comparison of numerical techniques for modeling electromagnetic dispersive media," *IEEE Microwave Guided Wave Lett.*, vol. 5, pp. 426–428, Dec. 1995.
[8] R. J. Luebbers, F. Hunsberger, K. S. Kunz, R. Standler, and M. Schneider, "A frequency-dependent finite-difference time-domain formulation for dispersive materials," *IEEE Trans. Electromagn. Compat.*, vol. 32, pp. 222–227, Aug. 1990.
[9] F. Hunsberger, R. Luebbers, and K. Kunz, "Finite-difference time-domain analysis of gyrotropic media—I: Magnetized plasma," *IEEE Trans. Antennas Propagat.*, vol. 40, pp. 1489–1495, Dec. 1992.
[10] R. M. Joseph, S. C. Hagness, and A. Taflove, "Direct time integration of Maxwell's equations in linear dispersive media with absorption for scattering and propagation of femtosecond electromagnetic pulses," *Opt. Lett.*, vol. 16, pp. 1412–1414, 1991.
[11] W. Liniger and R. A. Willoughby, "Efficient integration methods for stiff systems of ordinary differential equations," *SIAM J. Numer. Anal.*, vol. 7, pp. 47–66, 1970.
[12] J. L. Young, A. Kittichartphayak, Y. M. Kwok, and D. Sullivan, "On the dispersion errors related to $(\mathrm{FD})^2\mathrm{TD}$ type schemes," *IEEE Trans. Microwave Theory Tech.*, vol. 43, pp. 1902–1909, Aug. 1995.
[13] P. Petropoulos, "Stability and phase error analysis of FD-TD in dispersive dielectrics," *IEEE Trans. Antennas Propagat.*, vol. 42, pp. 62–69, Jan. 1994.
[14] Y. N. Taranenko, U. S. Inan, and T. F. Bell, "Interaction with the lower ionosphere of electromagnetic pulses from lightning: Heating, attachment, and ionization," *Geophys. Res. Lett.*, vol. 20, pp. 1539–1542, 1993.
[15] C. K. Birdsall and A. B. Langdon, *Plasma Physics Via Computer Simulation.* Bristol, England: Adam Hilger, 1991.
[16] T. Tajima, *Computational Plasma Physics: With Applications to Fusion and Astrophysics.* Redwood City, CA: Addison-Wesley, 1989.
[17] K. G. Budden, *The Propagation of Radio Waves.* New York: Cambridge Univ. Press, 1985.
[18] T. H. Stix, *Waves in Plasmas.* New York: American Inst. Phys., 1992.
[19] K. S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antennas Propagat.*, vol. 14, pp. 302–307, May 1966.
[20] R. Richtmyer and K. Morton, *Difference Methods for Initial-Value Problems.* New York: Wiley, 1967.
[21] B. Gustafsson, H.-O. Kreiss, and A. Sundström, "Stability theory of difference approximations for initial boundary value problems II," *Mathem. Computat.*, vol. 26, pp. 649–686, 1972.
[22] D. Potter, *Computational Physics* New York: Wiley, 1973.

Steven A. Cummer (S'91) received the B.S. (with distinction) and the M.S. degrees in electrical engineering from Stanford University, Stanford, CA, in 1991 and 1993, respectively. He is currently working toward the Ph.D. degree in electrical engineering at Stanford University.

His current research activities include modeling the propagation of VLF and ELF radiation from lightning discharges and investigating space weather effects on the lower ionosphere.

Mr. Cummer received an Outstanding Student Paper Award at the 1994 American Geophysical Union Fall Meeting and was a finalist in the Student Paper Competition at the 1995 IEEE Antennas and Propatation Society International Symposium.