A New Algorithm for Independent Component Analysis
With or Without Constraints
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Background

ICA Model

- The Mixing Equation
  \[ y(t) = A x(t) + n(t) \]
  (1)

  Where, \( x(t) \) is the independent source signals, \( y(t) \) is the mixed signals, \( A \) is the mixing matrix (with columns \( a_k \) = \( \{a_{k1}, a_{k2}, \ldots, a_{kn}\} \) corresponding to each source sample where \( k = 1, 2, \ldots, n \) and \( \{a_{k1}, a_{k2}, \ldots, a_{kn}\} \) is the \( k \)-th column vector of \( A \).

  \( n(t) \) is the noise vector.

  \- Objective: Assuming the signals in \( y(t) \) are independent, estimate \( A \) from \( y(t) \) (from here on).

Whitening - First Step of the Two-stage ICA

1. Do the whitening matrix

2. \( A \) is the whitened version.

3. Select non-white data:

4. Whitened ICA Model

5. The white Data

6. Whitening the data

7. \( A \) is the whitening matrix

Maximizing the JADE Criterion - Second Step of the Two-stage ICA

1. The JADE Criterion

2. \( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} a_{kj}^2 \right) \)

3. \( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} a_{kj}^2 \right) \)

4. \( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} a_{kj}^2 \right) \)

5. \( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} a_{kj}^2 \right) \)

6. \( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} a_{kj}^2 \right) \)

New Algorithm for ICA

The Joint Diagonalization Problem

\[ \mathbf{W}^T \mathbf{U} \mathbf{Q} \mathbf{U}^T \mathbf{W} = \mathbf{W}^T \mathbf{U} \mathbf{Q} \mathbf{U}^T \mathbf{W} \]
(5-A)

\[ \mathbf{Q} = \mathbf{U} \mathbf{W} \]
(5-B)

The Column-wise Processing

\[ \mathbf{w}_k = \text{argmax} \left( \sum_{i=1}^{n} \{|\mathbf{x}_i^T \mathbf{a}_k|\} \right) \]
(6-A)

\[ \mathbf{w}_k = \text{argmax} \left( \sum_{i=1}^{n} \{|\mathbf{x}_i^T \mathbf{a}_k|\} \right) \]
(6-B)

Holding of Constant (C-C) - Orthogonal Projection

Suppose we have found 1 column of \( \mathbf{U} \) and want to find the 2nd column.

Let \( \mathbf{U}_{(k)}, \mathbf{U}_{(k+1)} \), \( \mathbf{x} \).

Then, \( \mathbf{U}_{(k+1)} = \mathbf{U}_{(k)} \mathbf{U}_{(k)}^T \mathbf{x} \), \( \mathbf{U}_{(k+1)} \).

\[ \mathbf{U}_{(k+1)} = \mathbf{U}_{(k)} \mathbf{U}_{(k)}^T \mathbf{x} \]
(7-A)

\[ \mathbf{U}_{(k+1)} = \mathbf{U}_{(k)} \mathbf{U}_{(k)}^T \mathbf{x} \]
(7-B)

Thus, the dual problem of (5-C) can be formulated as

\[ \mathbf{w}_k = \text{argmax} \left( \sum_{i=1}^{n} \{|\mathbf{x}_i^T \mathbf{a}_k|\} \right) \]
(8-A)

\[ \mathbf{w}_k = \text{argmax} \left( \sum_{i=1}^{n} \{|\mathbf{x}_i^T \mathbf{a}_k|\} \right) \]
(8-B)

Symmetric Alternating-Signs-Algorithm (Symmetric ASA)

- The Lagrangian function (16-B)

- The Karush–Kuhn–Tucker (KKT) optimality conditions in

- with the matrix function \( \mathbf{F}(\mathbf{x}_i) \) defined as

- for any \( \mathbf{C} \) is clearly convex if \( \mathbf{F}(\mathbf{x}_i) \) is an increasing function from \( \mathbf{H}(1) \). (11)

- Then \( \mathbf{w}_k \) is the largest eigenvector of \( \mathbf{F}(\mathbf{x}_i) \) to maximize the objective function. We give the algorithm.

Algorithm 1 Symmetric Alternating-Signs-Algorithm (Symmetric ASA)

1. Initialze \( \mathbf{w}_k \) and \( \mathbf{w}_k \) \( \{ \mathbf{w}_k^0 \} \) and \( \{ \mathbf{w}_k^0 \} \) for \( k = 1, 2, \ldots, K \). \( \{ \mathbf{w}_k^0 \} \) and \( \{ \mathbf{w}_k^0 \} \) to 0.

2. Find the maximum eigenvalue, denoted \( \mathbf{w}_k \) and \( \{ \mathbf{w}_k \} \) and the associated eigenvectors with 0 and 2 indices.

3. Check convergence of \( \mathbf{w}_k^m \) and \( \{ \mathbf{w}_k^m \} \) go to 1.

4. Check satisfaction of the quadratic equality constraint. If \( \mathbf{w}_k^m \) and \( \{ \mathbf{w}_k^m \} \) are not satisfied, go to 2.

5. Update \( \mathbf{w}_k^m \) and \( \{ \mathbf{w}_k^m \} \).

6. Go to 2.

Convergence of Symmetric ASA

Proposition 1 For any non-zero and non-zero \( \mathbf{w}_k \), the equivalence function \( \{ \mathbf{w}_k \} \) defined by

- The Symmetric ASA algorithm converges, i.e., \( \{ \mathbf{w}_k^m \} \) \( \{ \mathbf{w}_k^m \} \) to \( \{ \mathbf{w}_k \} \) and \( \{ \mathbf{w}_k \} \) to \( \{ \mathbf{w}_k \} \) to 0.

- Theorem 1. Should be the largest eigenvalue of \( \mathbf{F}(\mathbf{x}_i) \) to maximize the objective function. We give the algorithm.

Algorithm 2 Symmetric Alternating-Signs-Algorithm (Symmetric ASA)

1. Define the convergence parameter \( \gamma \) and \( \epsilon \). Let \( \mathbf{w}_k \) and \( \{ \mathbf{w}_k \} \) be sufficiently large positive matrices.

2. Initialize \( \mathbf{w}_k \) and \( \{ \mathbf{w}_k \} \) to \( 0 \). Normalize \( \mathbf{w}_k \) and \( \{ \mathbf{w}_k \} \) to zero. Lo go to 3.

3. Find the maximum eigenvalue, denoted \( \mathbf{w}_k \) and \( \{ \mathbf{w}_k \} \) and the associated eigenvectors with 0 and 2 indices.

4. Find the maximum eigenvalue, denoted \( \mathbf{w}_k \) and \( \{ \mathbf{w}_k \} \) and the associated eigenvectors with 0 and 2 indices.

5. Check convergence of \( \mathbf{w}_k \) and \( \{ \mathbf{w}_k \} \) go to 1.

6. Check satisfaction of the quadratic equality constraint. If \( \mathbf{w}_k \) and \( \{ \mathbf{w}_k \} \) are not satisfied, go to 2.

7. Update \( \mathbf{w}_k \) and \( \{ \mathbf{w}_k \} \).

8. Go to 2.

Example Results

Example results are shown on various benchmark problems. We simulate a linear array with 32 taps and one-dimensional signal placed in the far field of the system. The inter-tap distance is the wavelength. The three-estimated linear source signals with directions of arrival (DOA) of \( 40^\circ, 30^\circ \), and \( 140^\circ \) are shown in Fig. 1(a). The estimated source and DOAs by the original JADE and our new ICA, are shown respectively, in Fig. 1(b) and (c). We show the estimation results. If we constrained ICA, by blocking the squared magnitude of the correlated term between the constraining moving vector and the columns of \( \mathbf{A} \) and \( \mathbf{B} \), which is equivalent to constraining the DOAs of sources between \( 180^\circ \) and \( 270^\circ \), one can be seen from the squared magnitude of the constrained response of the constraining moving vector shown in Fig. 1(b).

Conclusions

The algorithm is given as

\[ \text{Proposition 2 For a fixed } \alpha > 0 \text{ and } \beta, \text{ minimizing from any } \mathbf{w}_k \text{ with small } \text{Lemma, the eigenvalue sequence } \{ \mathbf{w}_k^m \} \text{ produced by the Symmetric ASA algorithm converges, i.e., } \{ \mathbf{w}_k^m \} \text{ to } 0. \text{ Furthermore, we assume the maximum eigenvalue of } \mathbf{A}(\mathbf{w}_k^m), \text{ and } \{ \mathbf{w}_k^m \} \text{ to be } 0, \text{ all else multiplicity of one, then there exist } \mathbf{w}_k^m \text{ and } \{ \mathbf{w}_k^m \} \text{ and } \{ \mathbf{w}_k^m \} \text{ to be } 0. \]


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