Signal reconstruction from accumulation of bispectral radial slices

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1 Introduction

This paper considers the problem of reconstructing signals from their noisy and randomly shifted observations. This problem occurs in a variety of applications including image transmission,1–2 image sequence analysis3 and radar.4 Conventional approaches to this problem require that the relative motion between consecutive measurements be compensated. In image sequence analysis, shift and rotation must be estimated to establish correspondence between different frames before filtering can be applied.3 Similar cases occur in radar signals where consecutive high resolution range profiles (HRRPs) with variable time delays are first aligned so that noncoherent averaging can be performed.4 The motion compensation or alignment of two signals can be done by locating their maximum cross correlation.2,4 If precise alignment is achieved, then a simple averaging of the aligned observations will yield a maximum likelihood (ML) estimate under the assumption of Gaussian noise.5 However, the alignment is a quite time consuming process and an inaccurate motion estimate may result when the signal-to-noise ratio (SNR) is low.

To obviate this problem, several authors have proposed bispectrum based approaches.1–3 The use of a bispectrum for this particular problem is motivated by the following three facts. First, the bispectrum domain is a high SNR domain due to its blindness to any noise with symmetric probability density function (pdf). Second, the shift invariance of bispectra enables a direct averaging in the bispectrum domain without the need to do motion compensation or alignment. And third, the bispectrum is invertible and enables a complete reconstruction of the deterministic signal apart from a shift.

Bispectrum based signal recovery other than for avoiding motion compensation has been investigated for a long time in a broad context6–10 and numerous reconstruction methods have been developed. Our approach is different from existing methods in the choice of slices in the bispectrum domain. Up to now only a single 45 deg slice or multiple parallel-to-axis slices have been used to recover signals. These algorithms either are prone to low signal-to-noise ratio (SNR) estimates or improve SNR of high frequency estimates only. We propose an algorithm that accumulates multiple bispectral radial slices and use the accumulation to recover both the Fourier phase and the magnitude of the signal. Major advantages of the new algorithm are the reduced estimation errors and improvement in SNR in the whole frequency band. A mathematical formulation of the algorithm is given and fast algorithms are presented along with other implementation issues. Performance is evaluated using both simulated data and real radar data. Experimental results show that the proposed algorithm compares favorably with existing methods in terms of mean squared reconstruction error and computational complexity.

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Bispectrum based signal recovery other than for avoiding motion compensation has been investigated for a long time in a broad context6–10 and numerous reconstruction methods have been developed. Our approach is different from existing methods in the choice of slices in the bispectrum domain. Up to now only a single 45 deg slice or multiple parallel-to-axis slices have been used to recover signal Fourier phases and amplitudes. For example, the Lii and Rosenblatt algorithm6 relies on the first off-axis slice on rectangular grids for the recovery of signal phase. Because only a single slice is used this algorithm is prone to low SNR results with bandlimited signals.10 The Bartelt-Lohmann-Wirnitzer (BLW) algorithm7 alleviate this problem by utilizing all slices parallel to the axis. Each slice provides an independent representation of the signal phase and magnitude so that an averaging mechanism is obtained, but unfortunately the averaging improves the SNR of high frequency estimates only, since the number of slices available decreases as the frequency index decreases. Moreover the BLW algorithm relies on the biased values on axis to initialize its recursion for magnitude recovery and is therefore not suitable for reconstructing deterministic signals from noise.5 This situation is improved in the nonrecursive least squares (LS) algorithm2,8,9 which uses all nond Reedant bispectral values to form an overdetermined set of simultaneous linear equations and yields an LS solution. Disadvantage of the LS algorithm may derive from the computational cost of matrix inversion. The only algorithm using a radial slice was developed by Dianat and Raghuveer.1 Their algorithm utilizes the 45 deg radial slice
of the bispectrum and thus obviate the problem with band-
limited signals. In addition, the Dianat-Raghuveer algo-
rithm lacks an averaging mechanism and suffers from high
variance. In this paper, we propose an algorithm that
accumulates multiple bispectral radial slices and uses the
resulting accumulation to recover both Fourier phase and
magnitude of the signal. Major advantages of using mul-
tiple radial slices over a single 45 deg slice or multiple
parallel-to-axis slices are the reduced estimation errors due
to the accumulation and potential of using all bispectral
values to improve the SNR in the whole frequency band.

The remainder of the paper is organized as follows. The
reconstruction problem is formulated in Sec. 2. The recon-
struction algorithm using accumulation of bispectral radial
slices is developed in Sec. 3. Fast algorithms and other
implementation issues are presented in Sec. 4. Performance
evaluation is made in Sec. 5. Finally, Sec. 6 concludes the
paper.

2 Formulation of the Reconstruction Problem

Consider a deterministic signal \( s(n) \) corrupted by additive
zero-mean, colored and stationary noise \( v(n) \) with a sym-
metric pdf. We have \( L \) noisy observations of the signal
\[
x_i(n) = s(n - n_i) + v_i(n)
\]  (1)
where \( n_i \) is the time delay of the \( i \)th observation and \( i = 0, 1, \ldots, L - 1 \). We aim at filtering out the noise and ob-
tain an estimate \( \hat{s}(n) \) as close as \( s(n) \) using the \( L \) noisy
observations. The bispectrum based approach to the prob-
lem usually take the following three steps:1–3

1. Compute the bispectrum \( B_{x_i}(\omega_1, \omega_2) \) of \( x_i(n) \) for
   \( i = 1, \ldots, L \).

2. Average \( B_{x_i}(\omega_1, \omega_2) \) over \( i \) to obtain \( B^{(L)}(\omega_1, \omega_2) \),
   estimated bispectrum of the true signal \( s(n) \) from the
   \( L \) noisy observations.

3. Reconstruct \( s(n) \) from its bispectrum estimate
   \( B^{(L)}(\omega_1, \omega_2) \).

It has been proven that \( \lim_{L \to \infty} B^{(L)}(\omega_1, \omega_2) \) is an
asymptotically unbiased estimate if \( s(n) \) is a zero mean
signal.2 For nonzero mean signals it was suggested that
bispectral values on the axis should be avoided, because these
estimates are biased.2

In practice, only a finite number of observations are available and therefore residue noises still exist in
\( B^{(L)}(\omega_1, \omega_2) \). Further improvement in the SNR of the re-
constructed signal \( \hat{s}(n) \) will rely on the reconstruction al-
gorithm.

3 Signal Reconstruction from Accumulation of
Bispectral Radial Slices

The bispectrum \( B(\omega_1, \omega_2) \) of a deterministic discrete signal
\( x(n) \) is defined as11
\[
B(\omega_1, \omega_2) = X(\omega_1)X(\omega_2)\hat{X}(\omega_1 + \omega_2), \tag{2}
\]
where \( X(\omega) \) is the Fourier transform of \( x(n) \) and \( \hat{\cdot} \) denotes
complex conjugate.

From Eq. (2) the two governing identities from which
derivation of most reconstruction algorithms proceeds fol-
lows
\[
\psi(\omega_1, \omega_2) = \phi(\omega_1) + \phi(\omega_2) - \phi(\omega_1 + \omega_2), \tag{3}
\]
\[
\tilde{B}(\omega_1, \omega_2) = \tilde{X}(\omega_1) + \tilde{X}(\omega_2) + \tilde{X}(\omega_1 + \omega_2), \tag{4}
\]
where \( \phi(\omega) \) and \( \psi(\omega_1, \omega_2) \) are the Fourier phase and
bispectral phase, respectively, and
\[
\tilde{X}(\omega) = \ln |X(\omega)|, \tag{5}
\]
\[
\tilde{B}(\omega_1, \omega_2) = \ln |B(\omega_1, \omega_2)|, \tag{6}
\]
are the Fourier log-magnitude and bispectral log-
magnitude, respectively.

3.1 Phase Reconstruction

It follows from Eq. (3) that
\[
\psi(\omega, m \omega) = \phi(\omega) + \phi(m \omega) - \phi(\omega + m \omega). \tag{7}
\]
By assigning integers 1, 2, \ldots, \( M - 1 \) to \( m \) in Eq. (7), we
obtain \( M - 1 \) radial slices in the bispectral phase domain.
Summing up these slices yields
\[
\psi^{(M)}(\omega) = \sum_{m=1}^{M-1} \psi(\omega, m \omega) = M \phi(\omega) - \phi(M \omega), \tag{8}
\]
where \( \psi^{(M)}(\omega) \) is the accumulation of radial slices in the
bispectral phase domain. Obviously \( \psi^{(2)}(\omega) \) is the 45 deg
slice and \( M \geq 2 \).

Because \( x(n) \) is a discrete signal, \( \phi(\omega) \) and \( \psi(\omega_1, \omega_2) \)
are periodic with periods \( 2\pi \) in their normalized frequen-
cies \( \omega \) and \( (\omega_1, \omega_2) \), respectively. Hence, \( \psi^{(M)}(\omega) \) is also
\( 2\pi \) periodic in \( \omega \). For \( x(n) \) of length \( N \), Eq. (8) can be
rewritten in the discrete forms as
\[
\psi^{(M)}_N(n) = \sum_{m=1}^{M-1} \psi(n, mn) = M \phi_N(n) - \phi_N(Mn), \tag{9}
\]
where
\[
\psi^{(M)}_N(n) = \psi^{(M)}(\omega)|_{\omega = 2\pi N / n}, \quad \psi(n, mn) = \psi(\omega, m \omega)|_{\omega = 2\pi N / n}, \quad \phi_N(n) = \phi(\omega)|_{\omega = 2\pi n / n}
\]
and \( n = 0, 1, \ldots, M - 1 \). Note the indices \( mn \) and \( Mn \) are taken as
modulo \( N \) because \( \phi(n) \) and \( \psi(n_1, n_2) \) are all periodic
with period \( N \).

It is easily seen that Eq. (9) defines a system of \( N \) simulta-
aneous linear equations in \( N \) unknowns \( \phi_N(0), \phi_N(1), \ldots, \phi_N(N - 1) \). Using matrix notation, we can rewrite Eq. (9) as
\[
\psi^{(M)}_N = A^{(M)}_N \phi_N \tag{10}
\]
with
\[ \Psi_N^{(M)} = [\psi_N^{(M)}(0), \psi_N^{(M)}(1), \ldots, \psi_N^{(M)}(N-1)]^T, \]  
\[ \Phi_N = [\phi_N(0), \phi_N(1), \ldots, \phi_N(N-1)]^T, \]  
and \( \mathbf{A}_N^{(M)} \) is an \( N \times N \) matrix, which can be decomposed as
\[ \mathbf{A}_N^{(M)} = \mathbf{M}_N - \mathbf{P}_N^{(M)}, \]  
where \( \mathbf{I}_N \) is the \( N \times N \) identity matrix, \( \mathbf{M}_N \) results from the first term of Eq. (9) and \( -\mathbf{P}_N^{(M)} \) is governed by the second term of Eq. (9). Note that the nonzero elements of \( \mathbf{P}_N^{(M)} \) are all 1’s and each row of \( \mathbf{P}_N^{(M)} \) has only one nonzero element.

It is readily seen from Eq. (13) that \( \mathbf{A}_N^{(M)} \) is diagonally dominant with respect to its rows because \( M \geq 2 \) and \( \mathbf{P}_N^{(M)} \) has only one nonzero element 1 appearing in each row. Thus, by virtue of the Gersgorin theorem, \( \mathbf{A}_N^{(M)} \) is of full rank and a unique solution for \( \phi_N \) in Eq. (10) is insured.

Using the discrete Fourier transform (DFT), we can have the following equivalent representation of Eq. (10)
\[ \Psi_N^{(M)} = \mathbf{A}_N^{(M)} \Phi_N, \]  
where
\[ \Psi_N^{(M)} = \mathbf{F} \psi_N^{(M)}, \]  
\[ \Phi_N = \mathbf{F} \phi_N, \]  
\[ \mathbf{A}_N^{(M)} = \mathbf{F} \mathbf{A}_N^{(M)} \mathbf{F}^{-1}. \]

and \( \mathbf{F} \) is the DFT matrix. We now prove that
\[ \mathbf{A}_N^{(M)} = \mathbf{A}_N^{(M)}^T. \]  
Denote
\[ \Psi_N^{(M)} = [\psi_N^{(M)}(0), \psi_N^{(M)}(1), \ldots, \psi_N^{(M)}(N-1)]^T, \]  
\[ \Phi_N = [\phi_N(0), \phi_N(1), \ldots, \phi_N(N-1)]^T. \]

Then Eq. (9) can be expanded as
\[ \sum_{k=0}^{N-1} \psi_N^{(M)}(k) \exp \left( \frac{2\pi i}{N} nk \right) = M \sum_{k=0}^{N-1} \phi_N(k) \exp \left( \frac{2\pi i}{N} nk \right) - \sum_{k=0}^{N-1} \phi_N(k) \exp \left( \frac{2\pi i}{N} nMk \right). \]  

By equating the coefficients of like terms on both sides of Eq. (21), we obtain for \( k=0,1,\ldots,N-1, \)
\[ \psi_N^{(M)}(k) = M \phi_N(k) - \sum_{j \in \Lambda_k} \phi_N \left( k + \frac{iN}{M} \right). \]  

with
\[ \Lambda_k = \left\{ i: 0 \leq i \leq M - 1, \frac{k + iN}{M} \text{ is an integer} \right\}. \]

The summation in Eq. (22) results from the fact that the period of \( \phi(n) \) is \( M \) times that of \( \phi(Mn) \).

Obviously, the linear equations defined by Eqs. (22) and (23) are identical to those defined by Eq. (14). Then \( \mathbf{A}_N^{(M)} \) is governed by Eqs. (22) and (23), from which it is seen that \( \mathbf{A}_N^{(M)} \) has the following decomposition
\[ \mathbf{A}_N^{(M)} = \mathbf{M}_N - \mathbf{P}_N^{(M)}, \]  
where \( \mathbf{M}_N \) is governed by the first term of Eq. (22), while \( -\mathbf{P}_N^{(M)} \) is governed by the summation of Eq. (22).

We now show that
\[ \mathbf{P}_N^{(M)} = \mathbf{P}_N^{(M)}^T. \]  
Let the elements of \( \mathbf{P}_N^{(M)} \) be denoted by \( p_{k,l} \), where \( k,l = 0,1,\ldots,N-1 \). It follows from Eq. (22) that
\[ p_{k,l} = \begin{cases} 1, & \text{if } k = lM \mod N \\ 0, & \text{else}. \end{cases} \]  
Similarly, let the elements of \( \mathbf{P}_N^{(M)} \) be denoted by \( p_{k,l} \), where \( k,l = 0,1,\ldots,N-1 \). It follows from Eq. (9) that
\[ p_{k,l} = \begin{cases} 1, & \text{if } l = kM \mod N \\ 0, & \text{else}. \end{cases} \]

By comparing of Eqs. (26) and (27), we see that Eq. (25) holds. Then, by Eqs. (13) and (24), Eq. (18) also holds.

Note from Eq. (9) that \( \psi_N^{(N+1)}(n) \) contain the bispectral values on axis that are biased' and \( \psi_N^{(M)}(n) \) with \( M > (N + 1) \) contains redundant slices because of the periodicity of \( \Phi_N(n_1, n_2) \). Therefore these accumulations should be avoided. In the rest of the paper we assume \( 2 \leq M \leq N \).

3.2 Magnitude Reconstruction
From Eqs. (4) to (6), the radial slices in the bispectral log-magnitude domain follows:
\[ \tilde{B}(\omega, m\omega) = \tilde{X}(\omega) + \tilde{X}(m\omega) + \tilde{X}(\omega + m\omega). \]  

Summing up these slices over \( 1 \leq m \leq M - 1 \) yields
\[ \tilde{B}^{(M)}(\omega) = \sum_{m=1}^{M-1} \tilde{B}(\omega, m\omega) \]
\[ = M\tilde{X}(\omega) + 2 \sum_{m=2}^{M-1} \tilde{X}(m\omega) + \tilde{X}(M\omega), \]  
where \( \tilde{B}^{(M)}(\omega) \) is the accumulation of radial slices in the bispectral log-magnitude domain. Obviously \( \tilde{B}^{(12)}(\omega) \) is the 45 deg slice. It is assumed that \( 2 \leq M \leq N \).
As with \( \phi(\omega), \phi(\omega_1, \omega_2) \) and \( \psi^{(M)}(\omega), X(\omega), \tilde{B}(\omega_1, \omega_2) \) and \( \tilde{B}^{(M)}(\omega) \) are periodic with period 2\( \pi \). For \( x(n) \) of length \( N \), we substitute the discrete frequencies \( \omega = (2\pi/N)n \) into Eq. (29) to yield its discrete forms

\[
\tilde{B}^{(M)}_N(n) = \sum_{m=1}^{M-1} \tilde{B}_N(m, mn) = M \tilde{X}_N(n) + 2 \sum_{m=2}^{M-1} \tilde{X}_N(mn) + \tilde{X}_N(Mn)
\]

where \( n = 0, 1, \ldots, N-1 \) and the indices \( mn \) and \( Mn \) are taken modulo \( N \).

Equation (30) defines a system of simultaneous linear equations, which, in matrix notation, can be written as

\[
\begin{align*}
\tilde{B}^{(M)}_N &= C^{(M)}_N \tilde{X}_N, \\
\end{align*}
\]

with

\[
\begin{align*}
\tilde{B}^{(M)}_N &= [\tilde{B}^{(M)}_N(0), \tilde{B}^{(M)}_N(1), \ldots, \tilde{B}^{(M)}_N(N-1)]^T, \\
\tilde{X}_N &= [\tilde{X}_N(0), \tilde{X}_N(1), \ldots, \tilde{X}_N(N-1)]^T,
\end{align*}
\]

and \( C^{(M)}_N \) in an \( N \times N \) matrix, which can be decomposed as

\[
C^{(M)}_N = M I_N + 2 \sum_{m=2}^{M-1} P^{(m)}_N + P^{(M)}_N,
\]

where \( I_N \) and \( P^{(M)}_N \) are the same as in Eq. (13), and \( P^{(m)}_N \) is \( P^{(m)}_N \) with \( M = m \). We see that \( M I_N, 2P^{(2)}_N, 2P^{(3)}_N, \ldots, 2P^{(M-1)}_N \) and \( P^{(M)}_N \) result from \( M \tilde{X}_N(n), \tilde{X}_N(2n), \tilde{X}_N(3n), \ldots, \tilde{X}_N[(M-1)n] \) and \( \tilde{X}_N(Mn) \) on the right-hand side of Eq. (30), respectively. Note \( C^{(M)}_N \) is a matrix with integer elements.

The rank of \( C^{(M)}_N \) can not be so easily determined as with \( A^{(M)}_N \). However, we have found by a manual examination up to \( N = 256 \) that \( C^{(M)}_N \) is full rank but in these two exceptional cases: (a) \( M = 3 \) and \( N \) is a multiple of any of the integers belonging to the set \( \{5, 19, 29, 43, 53, 67, 77, 91, 97, 101, 139, 149, 163, 173, 187, 197, 211, 221, 241\}; (b) \( M = 5 \) and \( N \) is a multiple of 217. With a careful examination, general conclusions about the singularity of \( C^{(M)}_N \) can also be drawn as follows:

1. If \( C^{(M)}_N \) is singular, \( C^{(M)}_{k,N} \) is also singular with \( k \) being an integer.
2. \( C^{(M)}_{N} \) will always be nonsingular when \( M \) is an even integer.
3. With large \( M \)s, only large \( N \)s may cause the singularity of \( C^{(M)}_N \).

Specifically, when \( M > 5 \), \( C^{(M)}_N \) will be nonsingular for \( N \)s up to \( N = 1024 \).

It is seen from these conclusions that the singularity of \( C^{(M)}_N \) can be avoided by choosing a not too small \( M \). In most practical applications \( N \) will not exceed 1024, and the singularity of \( C^{(M)}_N \) can be easily avoided by choosing \( M > 5 \). The full rank property of \( C^{(M)}_N \) ensures that Eq. (31) has a unique solution for \( \tilde{X}_N \). In those cases when \( C^{(M)}_N \) is singular, which are rarely encountered in practice and can be easily avoided, an LS solution results. Some underlying physical meanings of the singularity of \( C^{(M)}_N \) have been observed. These physical meanings are closely related to the properties of \( P^{(M)}_N \), which lead to the fast algorithms for phase reconstruction, therefore they will be discussed in Sec. 4.1 where the fast algorithms are presented.

Using DFT, we have the following equivalent representation of Eq. (31)

\[
\tilde{b}^{(M)}_N = c^{(M)}_N \tilde{x}_N,
\]

where

\[
\begin{align*}
\tilde{b}^{(M)}_N &= F \tilde{B}^{(M)}_N, \\
\tilde{x}_N &= F \tilde{X}_N, \\
c^{(M)}_N &= FC^{(M)}_N F^{-1},
\end{align*}
\]

and \( F \) is the DFT matrix. Using the same line of reasoning with \( a^{(M)}_N \) and \( A^{(M)}_N \), we can prove that

\[
c^{(M)}_N = C^{(M)}_N^T.
\]

4 Fast Algorithms and Other Implementation Issues

4.1 Fast Algorithms for Phase Reconstruction

Generally the phase and log-magnitude reconstruction algorithm can be implemented using Eqs. (10) and (31). Because \( A^{(M)}_N \) and \( C^{(M)}_N \) are full rank [we can change \( M \) by a small value to obtain a full rank when \( C^{(M)}_N \) is singular, which actually happens very rarely, as indicated in Sec. 3.2], these two equations have unique solutions

\[
\begin{align*}
\phi_N &= A^{(M)}_N^{-1} \psi^{(M)}_N, \\
\tilde{X}_N &= C^{(M)}_N^{-1} \tilde{B}^{(M)}_N,
\end{align*}
\]

for the phase and log-magnitude, respectively. The magnitude is then easily obtained from the log-magnitude using Eq. (5).

However, due to the special structure of \( A^{(M)}_N \), in many cases we need not solve for its inverse. By utilizing the properties of \( A^{(M)}_N \), fast algorithms can be found for Eq. (10) or Eq. (14) when \( N \) and \( M \) satisfy certain conditions.

From Eq. (13) we see that the structure of \( A^{(M)}_N \) is determined by \( P^{(M)}_N \). Now we show several properties of \( P^{(M)}_N \),

1. If \( N \) divides \( M \), say, \( N = LM \), then \( P^{(M)}_N \) has no non-zero elements except in its \( 0, M, \ldots, (L-1)M \)th
column. This can be proven as follows. Denote the elements of $\mathbf{P}_N^{(M)}$ by $p_{k,l}$, $k,l = 0,1,\ldots,N-1$. It follows from Eq. (27) that $p_{k,l}$ is a nonzero element if and only if $l = kM \mod N$. Because $N = LM$, the condition $l = kM \mod N$ is now reduced to $l = k \mod M$, or equivalently, $l = M \cdot (k \mod L)$. Therefore the column index $l$ of a nonzero element $p_{k,l}$ must be a multiple of $M$.

2. If $N$ and $M$ are coprime, each row and column of $\mathbf{P}_N^{(M)}$, respectively, has only one and one nonzero element. This can be verified as follows. Suppose column $i$ of $\mathbf{P}_N^{(M)}$ has two nonzero elements with row indices $n$ and $m$, it follows from Eq. (9) that $nM \mod N = mM \mod N = i$, where $0 \leq n,m \leq N-1$ and $n \neq m$. Then $(n-m)M$ will divide $N$. Since $0 < |n-m| \leq N-1$, this means $M$ divides $N$, which contradicts the assumption that $N$ and $M$ are coprime. Thus each column of $\mathbf{P}_N^{(M)}$ has one and only one nonzero element. The same property with rows follows originally from Eq. (9).

Property 1 implies that it is more convenient to solve Eq. (14) than to solve Eq. (10), since, to solve the former, we need actually to solve $L$ simultaneous equations in $L$ unknowns. Particularly, when $L = 1$ or $M = N$, no simultaneous equation needs to be solved. A solution for each unknown follows immediately from its corresponding equation.

Property 2 implies that in Eq. (10) or Eq. (14), each equation has at most two unknowns, and the equations are grouped into “rings.” The equations belonging to the same ring one depends on another consecutively and in a circular manner, that is, the first depends on the second, the second on the third, until the last depends on the very first equation. This means the first equation can be inserted into the second, the second into the third, until the last inserted into the first. In this way, a single equation in a single unknown is finally produced. Solving this equation to obtain the solution for the first unknown, then the second, the third, and up to the last equation, are solved consecutively.

Next we formulate several fast algorithms for phase reconstruction. Since computation of the bispectrum is a standard routine, and accumulation of the bispectral slices is straightforward, we omit these steps and assume $\psi_N^{(M)}$ are given and we aim at reconstructing $\phi_N$.

Phase Reconstruction Algorithm I (when $N = LM$)

1. Compute the DFT of $\psi_N^{(M)}$, denoted by $\Psi_N^{(M)}$.
2. $\Phi_N(i + jM) = 1/N \Psi_N^{(M)}(i + jM)$, where $i = 1,2,\ldots,M-1$ and $j = 0,1,\ldots,L-1$.
3. Solve the remaining $L$ simultaneous linear equations using any standard routine and obtain $\Phi_N(i)$ with $i = 0,M,\ldots,(L-1)M$.
4. Compute the IDFT of $\Phi_N^{(M)}$ to obtain $\phi_N^{(M)}$.

Phase Reconstruction Algorithm II (when $M = N$)

1. $\phi_N(0) = 1/(N-1) \psi_N^{(M)}(0)$
2. $\phi_N(i) = (1/N) [\psi_N^{(M)}(i) + \phi_N(0)]$ for $i = 1,2,\ldots,N-1$.

Phase Reconstruction Algorithm III (when $N$ and $M$ are coprime)

1. Label all equations of Eq. (10) as unsolved. Set $i = 0$. Assume $A_N^{(M)} = [a_{,ij}]_{i,j=0}$.
2. If equation $i$ has been labeled as solved, go to step 7.
3. If equation $i$ has only one unknown, let $\phi_N(i) = [\psi_N^{(M)}(i)]/a_{,ij}$, label equation $i$ as solved and go to step 7.
4. In this case equation $i$ starts a ring of equations, say, $i = i_0,i_1,\ldots,i_{K-1},i_K = i$, where equation $i$ has unknowns $\phi_N(i_1)$ and $\phi_N(i_1)$, equation $i_1$ has unknowns $\phi_N(i_1)$ and $\phi_N(i_2)$, and so on and equation $i_{K-1}$ has unknowns $\phi_N(i_{K-1})$ and $\phi_N(i)$. Let $\xi_0 = 1$ and $\theta_0 = 0$.
5. For $k = 1,2,\ldots,K$, do
   \begin{equation}
   \xi_k = \frac{a_{i_1-1,i_k-1}}{a_{i_1-1,i_k}},
   \end{equation}
   and
   \begin{equation}
   \theta_k = \frac{-a_{i_1-1,i_k} - \psi_N^{(M)}(i_k)}{a_{i_1-1,i_k}}.
   \end{equation}
6. $\phi_N(i) = \theta_K/(1 - \xi_K)$. Compute $\phi_N(i_1),\phi_N(i_2),\ldots,\phi_N(i_{K-1})$ by
   \begin{equation}
   \phi_N(i_k) = \xi_k \phi_N(i) + \theta_k.
   \end{equation}
   Label equation $i,i_1,\ldots,i_{K-1}$ as solved.
7. Set $i = i + 1$. If $i > N - 1$, stop the algorithm. Otherwise go to step 2.

It is easily seen that Algorithms II and III require only an order of $O(N)$ real multiplications for reconstructing a discrete signal of length $N$. To achieve the same reconstruction task, on the other hand, the LS algorithm must solve a set of $N^2/4 - [1 + (1)^{N-1}]/8$ linear equations for the least square solutions of $N-1$ unknowns,\(^8\) the number of real multiplications required by which is \(^3\) of the order of $O(N^3)$. The number of real multiplications required by Algorithm I for the same reconstruction task are estimated as

\[4N \log N\] incurred by two FFTs,
\[2(N-M)\] incurred by step 2,
\[O(N^3/M^3)\] incurred by solving the remaining $L$
\[-= N/M \] linear equations,
the summation of which is of the order of $O(N \log N)$ when $N/M$ keeps fixed or increases at a rate not exceeding the growth rate of $(N \log N)^{1/3}$.

The computation complexity of the three proposed algorithms, together with that of the LS algorithm for comparison, are summarized in Table 1. We see from Table 1 that the proposed algorithms are much more efficient in computation than the LS algorithm. Despite of their computational simplicity, the proposed algorithms display performance as good as that of the LS algorithm, as will be illustrated by the numerical results in Sec. 5.

**4.2 Revisiting the Singularity of Matrix $C_N^M$**

As seen from Eq. (34), the matrix $C_N^M$ can be decomposed as the sum of a scalar matrix $\text{diag}[M,M,\ldots,M]$ and $M-1$ matrices of the $P_N^M$ type, namely, $2P_N^N, 2P_N^{N-1}, \ldots, 2P_N^{N-M+1}$, and $P_N^M$. Suppose each of the $P_N^M$ type matrices has some of its rows grouped into rings, each of which is designated as an ordered integer sequence

$$ R_{N,j}^{(M)} = \{i_k : i_k = i_{k-1} \mod k_{N,j}^M \mod N \quad \text{and} \quad 0 \leq i_k \leq N-1, k = 0, 1, \ldots, k_{N,j}^M - 1\}, $$

where $i_k$ are interpreted as row indices. Note the coprime property of $N$ and $M$ does not necessarily hold here because we do not require all the rows are grouped into rings. Suppose further there exist rings $R_{N,j}^{(M_i)}, \ldots, R_{N,j}^{(M_M)}$, in each $P_N^M$ type matrix such that

$$ J_2 \cup R_{N,j}^{(M_2)} = \bigcup_{i=1}^{J_3} R_{N,j}^{(M_i)} = \cdots = \bigcup_{i=1}^{J_M} R_{N,j}^{(M_i)} = Q_{N,j}^{(M)}.$$

Take from $C_N^M$ all rows whose indices belong to $Q_{N,j}^{(M)}$ to form a submatrix $S_N^{(M)}$, and we refer to $S_N^{(M)}$ as a common union of rings (CUOR) of $C_N^M$. Obviously $C_N^M$ will be singular if it has a singular CUOR. It can be easily proven that $S_N^{(M)}$ has nonzero elements only in the columns whose indices belong to $Q_{N,j}^{(M)}$ and that each row and each nonzero column of $S_N^{(M)}$ has one and only one element contributed by each $P_N^M$ type matrix.

As an example, we take $N=5$ and $M=3$. Then we have

$$ C_5^{(3)} = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}, $$

and we can verify at a glance that $Q_5^{(3)} = \{1, 2, 3, 4\}$ and

$$ S_5^{(3)} = \begin{bmatrix} 0 & 3 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 2 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}. $$

A close examination reveals that the rings in $S_5^{(3)}$ are arranged in such a special manner that the nonzero elements coincide to cancel. Specifically, weighting rows 1, 2, 3 and 4 by $-1, 1, 1, -1$, respectively, and summing up the weighted rows cancels all nonzero elements and yields a zero row, thus leading to the singularity of $C_5^{(3)}$, and consequently, of $C_5^{(3)}$. A CUOR that has such a special arrangement of rings that its nonzero elements coincide to cancel is referred to as an effective CUOR. The phenomenon of effective CUORs leading to the singularity of $C_N^M$ does not stand alone for $N=5$ and $M=3$, but has been observed as the general underlying physical cause of the singularity of $C_N^M$.

It can be deduced that if $C_N^M$ has an effective CUOR $S_N^{(M)}$, then $C_N^{(M)}$ with an integer $k$ must have a corresponding effective CUOR $S_N^{(M)}$, the row and column indices of whose nonzero elements as they appear in $C_N^{(M)}$ are $k$ times the corresponding indices of nonzero elements of $S_N^{(M)}$ as they appear in $C_N^{(M)}$. The gives the physical meaning that underlies conclusion 1 in Sec. 3.2. Taken as an example, let $N=10$ and $M=3$. It can be easily found that $Q_{10}^{(3)} = \{2, 4, 6, 8\}$ and

$$ S_{10}^{(3)} = \begin{bmatrix} 0 & 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 3 & 0 \end{bmatrix}. $$

A comparison of $Q_{10}^{(3)}$ with $Q_5^{(3)}$ and $S_{10}^{(3)}$ with $S_5^{(3)}$, verifies the above conclusion.

An effective CUOR causes singularity of $C_N^M$ and therefore should be avoided. Fortunately, an effective CUOR rarely exists. It is seen that in each row and nonzero column of a CUOR $S_N^{(M)}$, there are one nonzero element $M$ contributed by the scalar matrix, $M-2$ nonzero element 2 contributed by $2P_N^{(2)}, 2P_N^{(3)}, \ldots, 2P_N^{(N-M)}$, respectively, and one nonzero element 1 contributed by $P_N^{(M)}$. For even $M$’s, it is not possible to cancel all nonzero elements of $S_N^{(M)}$ by weighting by 1 or $-1$ and summing up its rows or nonzero columns, because the nonzero elements in its each
row and nonzero column are all summed to be an odd integer. Since canceling nonzero elements of $S_N^{(M)}$ by weights other than 1 and $-1$ is out of the question due to its special structure, all $S_N^{(M)}$ matrices with an even $M$, and consequently $C_N^{(M)}$ with an even $M$, is generally nonsingular.

For large $M$s, there is a large number of $P_N^{(M)}$ type matrices and it is more difficult to find an effective CUOR for them. Increasing $N$ will increase the possibility for a CUOR to be created and its nonzero elements to be canceled. Therefore only large $N$s may cause the singularity of $C_N^{(M)}$ with large $M$s.

4.3 Phase Unwrapping

In practice, only the principal arguments of the bispectral phase are computed, which differ from the true phase by $2\pi k(n,m)$ with $k(n,m) \in \{-1, 0, 1\}$. Therefore unwrapping is required to yield a correct phase estimate using Eq. (10) or Eq. (14). Here we take an unwrapping method similar to the one proposed by Rangoussi and Giannakis:

1. Obtain the phase estimate $\hat{\phi}(n)$ using a phase reconstruction algorithm that needs no phase unwrapping, say, the BLW algorithm.
2. Substitute $\hat{\phi}(n)$ into Eq. (3) to compute $k(n,m)$, rounding the results to their nearest integers.
3. Unwrap the bispectral phase by adding $2\pi k(n,m)$ to its principal arguments.

4.4 Choice of $N$ and $M$

Because of periodicity of the bispectrum, the radial slices being accumulated may contain redundant bispectral values, or bispectral values on the axis, at some frequencies. The former is not tolerable when the redundant values repeatedly occur in noise dominated regions, and the latter is suggested to be avoided because these values are biased.

Denote by $\gcd(N,n)$ the greatest common divisor of $N$ and $n$. It can be verified that in order for $\psi_N^{(M)}(n)$ or $\tilde{B}_N^{(M)}(n)$, with $1 \leq n \leq N-1$ to accumulate nonredundant and off-axis bispectral values, it should be satisfied that

$$M \leq \min_{1 \leq n \leq N-1} \frac{N}{8\gcd(N,n)}. \quad (45)$$

For example, the following $(N,M)$ pairs satisfies Eq. (45): (8,2), (15,3), (25,5), (29,29). It is seen that a prime $N$ enables the largest $M$ and benefits the most from the accumulation. In fact $\psi_N^{(M)}(n)$ or $\tilde{B}_N^{(M)}(n)$ with prime $N$ takes all but on-axis bispectral values into accumulation and yet no values are accumulated more than once. For nonprime $N$, the upper bound for $M$ can be relaxed if we allow $\psi_N^{(M)}(n)$ and $\tilde{B}_N^{(M)}(n)$ at certain $n$s to accumulate redundant and on-axis bispectral values. For example, by allowing $\psi_{25}^{(M)}(n)$ and $\tilde{B}_{25}^{(M)}(n)$ to accumulate redundant and on-axis bispectral values at $n = 5, 10, 15, 20$, the allowable value of $M$ can be as great as 25.

5 Experimental Results

5.1 A Simulation Example

The true signal is chosen to be $s(n)$ with $s(0) = 1$, $s(1) = -7.5$, $s(2) = 9.56$, $s(3) = -3.36$ and $s(n) = 0$ for $4 \leq n \leq 17$. This signal happens to be a nonminimum phase signal with zeros at $[0.7, 0.8, 6]$.

The $i$’th observed signal $x_i(n)$ is simulated as

$$x_i(n) = s(n-n_i) + v_i(n),$$

where $n_i$ is a random integer shift and $v_i(n)$ is zero-mean colored Gaussian noise generated by passing white Gaussian noise through an AR model with transfer function $H(z) = 1/(1-0.9z^{-1})$. Various SNRs are considered. In all cases, bispectra of 200 frames of $x_i(n)$ are computed and averaged to yield the estimated bispectrum $\hat{B}_i(n,m)$ for $s(n)$. The signal $s(n)$ is then reconstructed from $\hat{B}_i(n,m)$ using various reconstruction algorithms. Twenty Monte Carlo runs are performed to achieve realization-independent results. The mean squared error (MSE) for each run is defined by

$$\text{MSE} = \frac{1}{16} \sum_{n=0}^{16} [\hat{s}(n)-s(n)]^2 \div \sum_{n=0}^{16} [s(n)]^2,$$

where $\hat{s}(n)$ is the reconstructed signal after properly shifted with respect to $s(n)$. Performances of the various reconstruction algorithms are then evaluated in terms of the MSE averaged over all 20 runs.

The proposed algorithm (with $M$ chosen to be 17), along with the Dianat-Raghuveer algorithm,1 the BLW algorithm,7 the LS algorithm2 and the noncoherent averaging algorithm8 for comparison, are implemented. The bispectral phases are unwrapped for the proposed algorithm using phase estimates by the BLW algorithm. The phase unwrapping method for the LS algorithm is as in Ref. 9. The MSE in decibels as a function of SNR (also in decibels) are plotted in Fig. 1 for the five algorithms.

We see that the proposed algorithm achieves the same good results as the LS algorithm and yet is computationally simpler due to its fast implementation scheme. Results achieved by the Dianat-Raghuveer algorithm degrade because it utilizes only a single radial slice of the bispectrum. The BLW algorithm exhibits large errors because it is a recursive algorithm and relies on the bispectral axis to provide initial values for the recursion.

The effect of different values of $M$ on performance of the proposed algorithm is demonstrated in Fig. 2. It is seen that at low SNRs, larger $M$, thus larger number of radial slices accumulated, reduces the reconstruction error evidently. This is because by increasing the number of nonredundant (note $N$ is taken as the prime integer 17 in this example) radial slices in accumulation the size of noise samples is increased, thus the residue noise in the averaged bispectrum is eliminated better and SNR of the phase and magnitude estimates improved. At high SNRs, residue noises in the averaged bispectrum are small and the algorithm yields good results for even small $M$. Thus the algorithm benefits from large $M$s especially in low SNR cases.
5.2 Application to Radar Target Feature Extraction

We now describe application of the proposed algorithm to the relaxation of aspect-dependence of HRRPs, which are widely used in radar target recognition. The traditional method to relax the aspect-dependence of HRRPs relies on noncoherent averaging, which first aligns the HRRPs by locating their maximum correlation and then averages the aligned HRRPs over many aspects.\(^4\) This method is computationally costing due to the time-consuming aligning process.

An HRRP \(x(n)\) can be modeled as

\[
x(n) = \sum_{i=1}^{L_n} \sigma_{ni}^2 + 2 \sum_{i=1}^{L_n} \sum_{k=1}^{L_n-i} \sigma_{ni} \sigma_{nk} \cos(\theta_{nik})
\]

where \(L_n\) is the number of scatterers in the \(n\)’th range cell; \(\sigma_{ni}\) and \(r_{ni}\) are the scattering strength and down range of the \(i\)’th scatterer in the cell, respectively; \(\lambda\) is the wavelength; and \(N\) is the number of range cells.

It is seen that the first term on the right-hand side of Eq. (46) is the autoterm representing the energy scattered from the scatterers while the second term is the cross-term induced by the interference between scatterers. Obviously the autoterm is a deterministic signal and the cross-term is a random noise process due to the random nature of \(\theta_{nik}\).

Furthermore, \(\theta_{nik}\) can be regarded as uniformly distributed in \([-\pi, \pi]\), and consequently the cross-term of Eq. (46) has a symmetric probability distribution function. Thus the HRRP model in Eq. (46) is in accordance with the signal model in Eq. (1) and by transforming the HRRP to the bispectrum domain we can suppress its cross-term and recover its autoterm, which carries the target’s discrimination information and can serve as target features for identification purpose.

Figure 3 shows six HRRPs of the full sized real aircraft “Cessna Citation S/I” measured uniformly over an aspect range of 3 deg. We see that they are shifted due to the motion of the target and they fluctuate with aspect change due to the interference between scatterers. Our aim is to eliminate the interference, i.e., the cross-term, and reconstruct the autoterm of the HRRP.

The five algorithms as in Sec. 5.1 are implemented with \(N = 71\) (\(M = 71\) for the proposed algorithm). Twenty runs
are performed and for each run 20, 35, 50, 65, 80, 100, respectively, HRRP frames are used for averaging. The mean of the reconstructed autoterms from 50 frames over all 20 runs is plotted in Fig. 4 for each of the five algorithms. We see that the proposed algorithm produces results almost as good as those by the noncoherent averaging algorithm though averaging is now performed in the bispectrum domain instead of in the range domain.

Since real data are used here no true autoterm is available. Therefore the MSE for each run is now computed as

$$MSE = \frac{\sum_{n=0}^{70} (\tilde{s}(n) - \bar{s}(n))^2}{\sum_{n=0}^{70} \bar{s}(n)^2},$$

where $\bar{s}(n)$ is the autoterm reconstructed by the noncoherent averaging algorithm (100 frames used) and $\tilde{s}(n)$ is the autoterm reconstructed by other algorithms after properly shifted with respect to $\bar{s}(n)$. The MSEs averaged over 20 runs are plotted in decibels as a function of number of frames in Fig. 5 for the four algorithms other than the noncoherent averaging algorithm. We see that the proposed algorithm outperforms the other three algorithms, including the LS algorithm.

The superior performance of the proposed algorithm over the LS algorithm in this particular example is not surprising. Note that in this example, $M = N = 71$, which is the situation of Algorithm II. Hence, in reconstructing phase at discrete frequency $\omega_k = (2\pi/N)k$, the proposed algorithm use only the bispectral radial slice values at $\omega_k$ and $\omega_0$, whereas the LS algorithm uses all nonredundant bispectral values in reconstructing phase at $\omega_k$. Similar cases exist in amplitude reconstruction. Since cross-terms usually display flat spectra while auto-terms do not, there exist frequencies, say, $\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_k}$, where the autoterm to cross-term ratio (ACR) is low. The LS algorithm include these low ACR bispectral values in reconstruction at all frequencies, whereas the proposed algorithm exclude these values in reconstruction of signal at frequencies other than $\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_k}$. The inclusion of low ACR values in reconstruction at all frequencies is responsible for the degraded performance of the LS algorithm.

The MSEs of the proposed algorithm with different $M$ are plotted in decibels as a function of number of frames in Fig. 6. It is seen that on the whole increasing the number of accumulated slices improves the reconstruction accuracy. However it may sometimes happen that some intermediate $M$ performs better than larger $M$. This happens, for example, when such a $M$ selects bispectral values from the bispectral region where the autoterm dominates the cross-term.
6 Conclusions

We developed an algorithm using multiple radial slices of bispectrum for reconstructing deterministic signals from their noisy and randomly shifted observations. The multiple radial slices are accumulated to eliminate the residue noise in the averaged bispectrum and improve the SNR of the phase and magnitude estimates in the whole frequency band. The new algorithm is nonrecursive and does not rely on bispectral values on the axis, so it is particularly suitable for recovering deterministic signals from noises. Fast implementation schemes are presented for phase reconstruction that avoid matrix manipulation when the data length and number of slices satisfy certain conditions. With the data length chosen to be a prime integer the algorithm accumulates all possible non-redundant and off-axis bispectral values and benefits the most from the accumulation. Cases exist where some intermediate number of slices choose bispectral values from signal nominated bispectral regions and yield better results. The algorithm is evaluated using both simulated data and real radar data. The experimental results show that the proposed algorithm compares favorably with existing methods in terms of mean squared reconstruction error and computational complexity.

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