Midterm Solutions

EE 640
Stochastic Systems

Problem 1: Schwartz Inequality

Y

$\alpha() + w$

X

$$\lambda = \frac{\alpha E\{XY\}}{E\{Y^2\}} = \frac{\alpha^2 E\{X^2\}}{\alpha^2 E\{X^2\} + \frac{1}{\beta} E\{X^2\}} = \frac{\alpha^2}{\alpha^2 + \frac{1}{\beta}}$$

$$\frac{E\left\{(X - \hat{X})^2\right\}}{E\{X^2\}} \geq \frac{E\{X^2\} - \frac{E^2\{XY/\alpha\}}{E\{Y^2/\alpha^2\}}}{E\{X^2\}} = 1 - \frac{\alpha^2}{\alpha^2 + \frac{1}{\beta}} = \frac{\frac{1}{\beta}}{\alpha^2 + \frac{1}{\beta}}$$

$$\lambda = \frac{\alpha E\{XY\}}{E\{Y^2\}} \leq \frac{\alpha \sqrt{E\{X^2\} E\{Y^2\}}}{E\{Y^2\}} = \alpha \sqrt{\frac{E\{X^2\}}{E\{Y^2\}}} = \sqrt{\frac{\alpha^2}{\alpha^2 + \frac{1}{\beta}}}$$

HW Based on

2.22

$$0 \le E\{(X+\lambda Y)^2\} = E\{X^2\} + 2\lambda E\{XY\} + \lambda^2 E\{Y^2\}$$

$$let\ \lambda = -\frac{E\{XY\}}{E\{Y^2\}}, then$$

$$0 \le E\{X^2\} - \frac{E^2\{XY\}}{E\{Y^2\}} \Rightarrow E^2\{XY\} \le E\{X^2\}E\{Y^2\}$$

$$As\ 2E\{XY\} \le 2\sqrt{E\{X^2\}E\{Y^2\}}, \quad we\ have$$

$$E\{(X+Y)^2\} \le \left[\sqrt{E\{X^2\}} + \sqrt{E\{Y^2\}}\right]^2$$

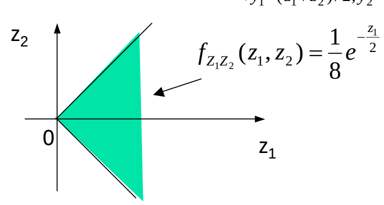
$$\sqrt{E\{(X+Y)^2\}} \le \sqrt{E\{X^2\}} + \sqrt{E\{Y^2\}}$$

Problem 2: Function of Random Variables

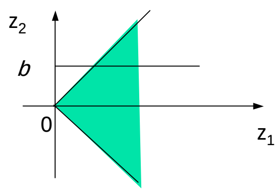
$$For\ y = \sum_{i=1}^{n} x_i^2, n = 2, \quad f_Y(y) = \frac{1}{2\Gamma(1)} e^{-\frac{y}{2}} = \frac{1}{2} e^{-\frac{y}{2}}, \quad f_{Y_1,Y_2}(y_1, y_2) = \frac{1}{4} e^{-\frac{y_1+y_2}{2}}$$

$Z_1 = Y_1 + Y_2$, $Z_2 = Y_1 - Y_2$; $Y_1 = (Z_1+Z_2)/2$, $Y_2 = (Z_1-Z_2)/2$

$$f_{Z_1Z_2}(z_1, z_2) = \left.\frac{f_{Y_1Y_2}(y_1, y_2)}{|J(y_1, y_2)|}\right|_{y_1=(z_1+z_2)/2, y_2=(z_1-z_2)/2} = \left.\frac{1}{8} e^{-\frac{z_1}{2}}\right|_{0 \le (z_1+z_2), 0 \le z_1 - z_2, z_1 \ge 0}$$



Problem 2: Function of Random Variables



$$f_{Z_2}(z_2) = \int_{|z_2|}^{\infty} f_{Z_1 Z_2}(z_1, z_2) dz_1 = \int_{|z_2|}^{\infty} \frac{1}{8} e^{-\frac{z_1}{2}} dz_1 = \frac{1}{4} e^{-\frac{|z_2|}{2}}$$

$$f_{Z_1|Z_2}(z_1 \mid z_2) = \frac{1}{2} e^{-\frac{z_1 - |z_2|}{2}}, \quad f_{Z_1|Z_2}(z_1 \mid z_2 = b) = \frac{1}{2} e^{\frac{|b|}{2}} e^{-\frac{z_1}{2}}$$

$$f_X(x) \sim e^{-x} \quad x > 0$$

$$Tchebycheff: P(X \geq a) \leq \frac{2}{a^2} \quad Chernoff: P(X > a) \leq a e^{1-a}, \; for\ a > 0$$

$$f_{Z_1}(z_1) \sim c e^{-\frac{z_1 - |b|}{2}} \quad where\ z_1 > |b|, c = \frac{1}{2}$$

$$Tchebycheff: P(Z_1 \geq a) = P\left(\frac{Z_1 - |b|}{2} \geq \frac{a - |b|}{2}\right) \leq \frac{2}{\left(\frac{a-|b|}{2}\right)^2} \quad Chernoff: P(Z_1 > a) \leq \left(\frac{a-|b|}{2}\right) e^{1-\left(\frac{a-|b|}{2}\right)}, \; for\ a > |b|$$

HW Based on

2.40

chi – square distribution

$$Y = X^2 \sim f_Y(y) = \frac{1}{2^{\frac{1}{2}} \pi^{\frac{1}{2}}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}$$

$$\psi_Y(j\omega) = (1 - 2j\omega)^{-1/2}$$

$$Y = \sum_{i=1}^{n} X_i^2$$

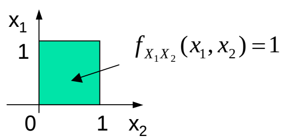
$$\psi_Y(j\omega) = (1 - 2j\omega)^{-n/2} \Leftrightarrow f_Y(y) = \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} e^{-\frac{y}{2}}$$

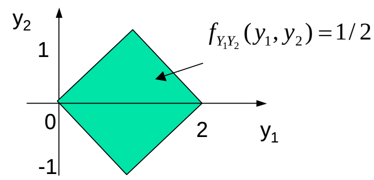
HW Based on

2.38 (a)

$Y_1=X_1 + X_2$, $Y_2 = X_1 - X_2$; $X_1 = (Y_1+Y_2)/2$, $X_2 = (Y_1-Y_2)/2$

$$f_{Y_1Y_2}(y_1,y_2) = \left.\frac{f_{X_1X_2}(x_1,x_2)}{|J(x_1,x_2)|}\right|_{x_1=(y_1+y_2)/2,\, x_2=(y_1-y_2)/2} = \left.\frac{1}{2}\right|_{0\le(y_1+y_2)\le 2,\, 0\le(y_1-y_2)\le 2}$$





HW Based on

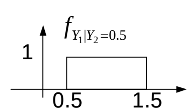
2.38 (b)

$$\rho_{y_1y_2} = \frac{E\{Y_1Y_2\}-\mu_{y_1}\mu_{y2}}{\sigma_{y_1}\sigma_{y2}} = \frac{E\{X_1^2-X_2^2\}-E\{X_1+X_2\}E\{X_1-X_2\}}{\sigma_{y_1}\sigma_{y2}}$$

$$= \frac{E\{X_1^2-X_2^2\}-[E^2\{X_1\}-E^2\{X_2\}]}{\sigma_{y_1}\sigma_{y2}} = 0$$

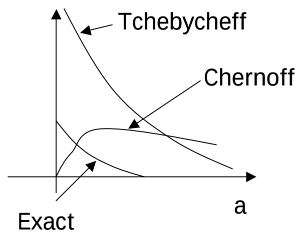
Doesn't imply independence

$E\{Y_1 \mid Y_2 = 0.5\} = 1$



HW Based on

2.49 (b)



$$f_X(x) \sim e^{-x} \quad x>0$$

$$Exact: P(X \geq a) = \int_a^\infty e^{-x}dx = e^{-a} \quad a>0$$

$$Tchebycheff: P(|X| \geq a) \leq \frac{1}{a^2}E\{X^2\} = \frac{1}{a^2}\int_0^\infty x^2 e^{-x}dx = \frac{2}{a^2}$$

$$Chernoff: P(X>a) \leq \min e^{-at}E\{e^{tx}\}$$

$$E\{e^{tx}\} = \int_0^\infty e^{tx}e^{-x}dx = \frac{1}{1-t}$$

$$let\ g(t) = \frac{e^{-at}}{1-t},\ then\ g'(t)=0 \Rightarrow (1-t)(-a)+1=0 \Rightarrow t=\frac{a-1}{a}$$

$$P(X>a) \leq ae^{1-a},\ for\ a>0$$

Problem 3: Byes Rules

P(Bull|increase)=P(increase|Bull)P(Bull)/[P(increase|Bull)P(Bull)+P(increase|Bear)P(Bear)] = 0.6x0.5/(0.6x0.5+0.1x0.5)=6/7 P(Bull)=P(Bear)=0.5

P(Bear|increase)=P(increase|Bear)P(Bear)/[P(increase|Bull)P(Bull)+P(increase|Bear)P(Bear)]= 0.1x0.5/(0.6x0.5+0.1x0.5)=1/7

P(Bull->Bear|decrease)=P(decrease|Bull->Bear)P(Bull->Bear)/
[P(decrease|Bull->Bear)P(Bull->Bear)+P(decrease|Bull->Bull)P(Bull->Bull)]
=(0.6x0.4)/(0.6x0.4+0.6x0.1)=4/5

P(Bear, Bear|decrease, decrease)=P(decrease, decrease|Bear, Bear)/
[P(decrease, decrease|Bear, Bear)+P(decrease, decrease|Bull, Bear)
+P(decrease, decrease|Bear, Bull)+P(decrease, decrease|Bull, Bull)]
= P(decrease, decrease|Bear->Bear) P(Bear->Bear)P(Bear)/
[P(decrease, decrease|Bear->Bear) P(Bear->Bear) P(Bear)
+P(decrease, decrease|Bull->Bear) P(Bull->Bear) P(Bull)
+P(decrease, decrease|Bear->Bull) P(Bear->Bull) P(Bear)
+P(decrease, decrease|Bull->Bull) P(Bull->Bull) P(Bull)]
= 0.6x0.6x0.6x0.5/[0.6x0.6x0.6x0.5+0.6x0.1x0.4x0.5+0.1x0.6x0.4x0.5+0.1x0.1x0.6x0.5]= 4/5

HW Based on

2.7 [7R | 3B] [4R | 5B]

A1: Transferred marble is red, P(A1) = 7/10

A2: Transferred marble is blue, P(A2) = 3/10

P(red|A1) = 5/10, P(red|A2) = 4/10

P(A2|red) = P(red|A2)P(A2)/P(red)

= (4/10x3/10)/(4/10x3/10+5/10x7/10)

2.9 Using Bayes rule

P(p) = P(p|s)P(s) + P(p|s^o)P(s^o)

P(s|p) = P(p|s)P(s)/P(p) = (0.9x0.75)/(0.9x0.75+0.2x0.25)