Midterm II Solutions

EE 640
Stochastic Systems

Problem 1: Schwartz Inequality

Y

$\alpha() + w$

X

$$\lambda = \frac{\alpha E\{XY\}}{E\{Y^2\}} = \frac{\alpha E\{\alpha X^2 + XW\}}{E\{\alpha^2 X^2\} + E\{2\rho\alpha X^2\} + \beta E\{X^2\}} = \frac{\alpha(\alpha+\rho)}{\alpha^2 + 2\rho\alpha + \beta}$$

$$\lambda = \frac{\alpha E\{XY\}}{E\{Y^2\}} \le \frac{\alpha\sqrt{E\{X^2\}E\{Y^2\}}}{E\{Y^2\}} = \alpha\sqrt{\frac{E\{X^2\}}{E\{Y^2\}}} = \sqrt{\frac{\alpha^2}{\alpha^2 + 2\rho\alpha + \beta}}$$

$$\frac{E\left\{(X - \hat{X})^2\right\}}{E\{X^2\}} \ge \frac{E\{X^2\} - \dfrac{E^2\{XY/\alpha\}}{E\{Y^2/\alpha^2\}}}{E\{X^2\}} = 1 - \frac{(\alpha+\rho)^2}{\alpha^2 + 2\rho\alpha + \beta} = \frac{\beta - \rho^2}{\alpha^2 + 2\rho\alpha + \beta}$$

HW Based on

2.22

$$0 \le E\{(X+\lambda Y)^2\} = E\{X^2\} + 2\lambda E\{XY\} + \lambda^2 E\{Y^2\}$$

$$let\ \lambda = -\frac{E\{XY\}}{E\{Y^2\}}, then$$

$$0 \le E\{X^2\} - \frac{E^2\{XY\}}{E\{Y^2\}} \Rightarrow E^2\{XY\} \le E\{X^2\}E\{Y^2\}$$

$$As\ 2E\{XY\} \le 2\sqrt{E\{X^2\}E\{Y^2\}}, \quad we\ have$$

$$E\{(X+Y)^2\} \le \left[\sqrt{E\{X^2\}} + \sqrt{E\{Y^2\}}\right]^2$$

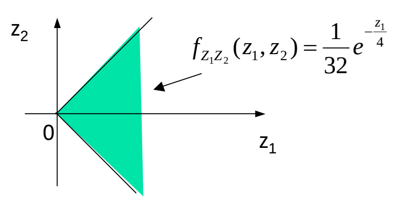
$$\sqrt{E\{(X+Y)^2\}} \le \sqrt{E\{X^2\}} + \sqrt{E\{Y^2\}}$$

Problem 2: Function of Random Variables

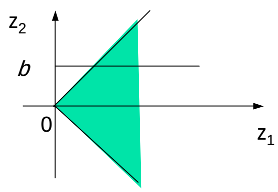
$$X \sim N(0,2) \quad For\ y = \sum_{i=1}^{n} x_i^2, n=2, \quad f_Y(y) = \frac{1}{4\Gamma(1)} e^{-\frac{y}{4}} = \frac{1}{4} e^{-\frac{y}{4}}, \quad f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{16} e^{-\frac{y_1+y_2}{4}}$$

$Z_1 = Y_1 + Y_2$, $Z_2 = Y_1 - Y_2$; $Y_1 = (Z_1+Z_2)/2$, $Y_2 = (Z_1-Z_2)/2$

$$f_{Z_1Z_2}(z_1,z_2) = \left.\frac{f_{Y_1Y_2}(y_1,y_2)}{|J(y_1,y_2)|}\right|_{y_1=(z_1+z_2)/2,\, y_2=(z_1-z_2)/2} = \left.\frac{1}{32} e^{-\frac{z_1}{4}}\right|_{0\le(z_1+z_2),\, 0\le z_1-z_2,\, z_1\ge 0}$$



Problem 2: Function of Random Variables



$$f_{Z_2}(z_2)=\int_{|z_2|}^{\infty} f_{Z_1Z_2}(z_1,z_2)dz_1=\int_{|z_2|}^{\infty}\frac{1}{32}e^{-\frac{z_1}{4}}dz_1=\frac{1}{8}e^{-\frac{|z_2|}{4}}$$

$$f_{Z_1|Z_2}(z_1\,|\,z_2)=\frac{1}{4}e^{-\frac{z_1-|z_2|}{4}},\quad f_{Z_1|Z_2}(z_1\,|\,z_2=b)=\frac{1}{4}e^{-\frac{z_1-|b|}{4}}$$

$$f_X(x)\sim e^{-x}\quad x>0\quad Tchebycheff: P(X>a)\le\frac{2}{a^2}\quad Chernoff: P(X>a)\le ae^{1-a},\ for\ a>0$$

$$f_{Z_2}(z_2)\sim\frac{1}{8}e^{-\frac{|z_2|}{4}}\quad where\ |z_2|>0,\ Tchebycheff: P(|Z_2|>b)=P\left(\frac{Z_2}{4}>\frac{b}{4}\right)\le\frac{2}{\left(\frac{b}{4}\right)^2}\quad Chernoff: P(|Z_2|>b)\le\left(\frac{b}{4}\right)e^{1-\left(\frac{b}{4}\right)}$$

$$Tchebycheff: \frac{1}{1-P(|Z_2|>b)}\le\frac{1}{1-\frac{2}{\left(\frac{b}{4}\right)^2}}\quad Chernoff: \frac{1}{1-P(|Z_2|>b)}\le\frac{1}{1-\left(\frac{b}{4}\right)e^{1-\left(\frac{b}{4}\right)}}$$

Problem 2: Function of random variables

$$f_{Z_1|Z_2}(z_1\,|\,t)\sim\frac{1}{4}e^{-\frac{z_1-|t|}{4}}\quad where\ z_1>|t|$$

$$Tchebycheff: P(Z_1>a,|Z_2|<b)=2\int_0^b P(Z_1\ge a\,|\,Z_2=t)\,f_{Z_2}(t)dt\le\frac{1}{2}\int_0^b\frac{1}{\left(\frac{a-t}{4}\right)^2}e^{-\frac{t}{4}}dt$$

$$<8\int_0^b\frac{1}{(a-t)^2}dt=8\int_{a-b}^{a}\frac{1}{x^2}dx=\frac{8b}{(a-b)a}$$

$$Chernoff: P(Z_1>a,|Z_2|<b)\le\frac{1}{4}\int_0^b\left(\frac{a-t}{4}\right)e^{1-\left(\frac{a-t}{4}\right)}e^{-\frac{t}{4}}dt=\frac{1}{16}e^{1-\frac{a}{4}}\int_{a-b}^{a}xdx=\frac{(2a-b)b}{32}e^{1-\frac{a}{4}}$$

$$P(Z_1>a\,|\,|Z_2|<b)=\frac{P(Z_1>a,|Z_2|<b)}{P(|Z_2|<b)}=\frac{P(Z_1>a,|Z_2|<b)}{1-P(|Z_2|>b)}$$

HW Based on

2.40

$$\text{chi} - \text{square distribution}$$

$$Y = X^2 \sim f_Y(y) = \frac{1}{2^{\frac{1}{2}}\pi^{\frac{1}{2}}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}$$

$$\psi_Y(j\omega) = (1-2j\omega)^{-1/2}$$

$$Y = \sum_{i=1}^{n} X_i^2$$

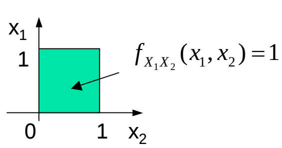
$$\psi_Y(j\omega) = (1-2j\omega)^{-n/2} \Leftrightarrow f_Y(y) = \frac{1}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} e^{-\frac{y}{2}}$$

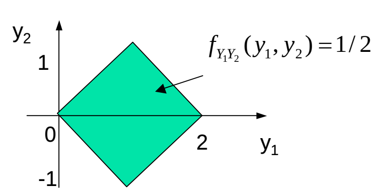
HW Based on

2.38 (a)

$Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$; $X_1 = (Y_1+Y_2)/2$, $X_2 = (Y_1-Y_2)/2$

$$f_{Y_1Y_2}(y_1,y_2) = \left.\frac{f_{X_1X_2}(x_1,x_2)}{|J(x_1,x_2)|}\right|_{x_1=(y_1+y_2)/2,\, x_2=(y_1-y_2)/2} = \left.\frac{1}{2}\right|_{0\le(y_1+y_2)\le 2,\, 0\le(y_1-y_2)\le 2}$$





HW Based on

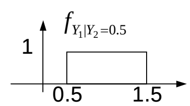
2.38 (b)

$$\rho_{y_1y_2}=\frac{E\{Y_1Y_2\}-\mu_{y_1}\mu_{y2}}{\sigma_{y_1}\sigma_{y2}}=\frac{E\{X_1^2-X_2^2\}-E\{X_1+X_2\}E\{X_1-X_2\}}{\sigma_{y_1}\sigma_{y2}}$$

$$=\frac{E\{X_1^2-X_2^2\}-[E^2\{X_1\}-E^2\{X_2\}]}{\sigma_{y_1}\sigma_{y2}}=0$$

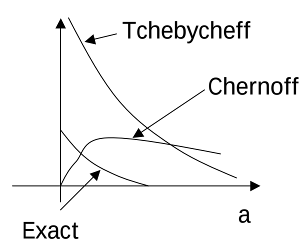
Doesn't imply independence

$$E\{Y_1\mid Y_2=0.5\}=1$$



HW Based on

2.49 (b)



$$f_X(x)\sim e^{-x}\quad x>0$$

$$Exact: P(X\geq a)=\int_a^\infty e^{-x}dx=e^{-a}\quad a>0$$

$$Tchebycheff: P(|X|\geq a)\leq\frac{1}{a^2}E\{X^2\}=\frac{1}{a^2}\int_0^\infty x^2e^{-x}dx=\frac{2}{a^2}$$

$$Chernoff: P(X>a)\leq\min e^{-at}E\{e^{tx}\}$$

$$E\{e^{tx}\}=\int_0^\infty e^{tx}e^{-x}dx=\frac{1}{1-t}$$

$$let\ g(t)=\frac{e^{-at}}{1-t},\ then\ g'(t)=0\Rightarrow(1-t)(-a)+1=0\Rightarrow t=\frac{a-1}{a}$$

$$P(X>a)\leq ae^{1-a},\ for\ a>0$$

Problem 3: Byes Rules

[P(Bull) P(Bear)]=[4/7 3/7]

P(Bull|decrease)=2/11 P(Bear|decrease) = 9/11

P(Bull->Bear|increase) = 1/15

P(Bull, Bull|flat, flat)=2/5

HW Based on

2.7 [7R | 3B] [4R | 5B]
A1: Transferred marble is red, P(A1) = 7/10
A2: Transferred marble is blue, P(A2) = 3/10
P(red|A1) = 5/10, P(red|A2) = 4/10
P(A2|red) = P(red|A2)P(A2)/P(red)
= (4/10x3/10)/(4/10x3/10+5/10x7/10)

2.9 Using Bayes rule
P(p) = P(p|s)P(s) + P(p|s^o)P(s^o)
P(s|p) = P(p|s)P(s)/P(p) = (0.9x0.75)/(0.9x0.75+0.2x0.25)