Homework Solutions

EE 640
Stochastic Systems

HW1A

2.1 (a) $P(A_1 \cap A_2 \cap A_3 \cap A_4) = 4/52 X 4/52 X 4/52 X 4/52$
(b) $P(A_1 \cap A_2 \cap A_3 \cap A_4) = 4/52 X 3/51 X 2/50 X 1/49$

2.3 (a) 20 outcomes
(R1, R2) (R1, R3) (R1, R4) (R1, R5)
(R2, R1) (R2, R3) (R2, R4) (R2, R5)
(R3, R1) (R3, R2) (R3, R4) (R3, R5)
(R4, R1) (R4, R2) (R4, R3) (R4, R5)
(R5, R1) (R5, R2) (R5, R3) (R5, R4)

HW1A

(b) $P(AB) = P(A|B)P(B) = 2/4 \times 3/5$

$A \in \{R1, R2, R3\}$, $B \in \{R1, R2, R3\}$

(c) $P(AB \cup BA) = P(A|B)P(B) + P(B|A)P(A)$

$= 3/4 \times 2/5 + 2/9 \times 3/5$

$A \in \{R1, R2, R3\}$, $B \in \{R4, R5\}$

(d) $P(AB) = P(A|B)P(B) = 3/4 \times 2/5 = 6/20$

$A \in \{R1, R2, R3\}$, $B \in \{R4, R5\}$

HW1B

2.7 [7R | 3B] [4R | 5B]

A1: Transferred marble is red, $P(A1) = 7/10$

A2: Transferred marble is blue, $P(A2) = 3/10$

$P(red|A1) = 5/10$, $P(red|A2) = 4/10$

$P(A2|red) = P(red|A2)P(A2)/P(red)$

$= (4/10 \times 3/10)/(4/10 \times 3/10 + 5/10 \times 7/10)$

2.9 Using Bayes rule

$P(p) = P(p|s)P(s) + P(p|s^o)P(s^o)$

$P(s|p) = P(p|s)P(s)/P(p) = (0.9 \times 0.75)/(0.9 \times 0.75 + 0.2 \times 0.25)$

HW1B

2.12

P(x1=1, x2=1, x3=1) = 3!/(1!1!1!)(1/2)^1(1/4)^1(1/4)^1

HW2A

2.14

$$E\{X\}=\sum_{k=0}^{\infty}k\frac{\lambda^k e^{-\lambda}}{k!}=\sum_{k=1}^{\infty}\frac{\lambda^k e^{-\lambda}}{(k-1)!}=\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}e^{-\lambda}}{(k-1)!}=\lambda\sum_{k'=0}^{\infty}\frac{\lambda^{k'}e^{-\lambda}}{k'!}=\lambda$$

$$E\{X^2\}=\sum_{k=0}^{\infty}k^2\frac{\lambda^k e^{-\lambda}}{k!}=\sum_{k=1}^{\infty}k\frac{\lambda^k e^{-\lambda}}{(k-1)!}=\sum_{k=2}^{\infty}\frac{\lambda^k e^{-\lambda}}{(k-2)!}+\sum_{k=1}^{\infty}\frac{\lambda^k e^{-\lambda}}{(k-1)!}$$

$$=\lambda^2\sum_{k=2}^{\infty}\frac{\lambda^{k-2}e^{-\lambda}}{(k-2)!}+\lambda=\lambda^2+\lambda$$

$$\sigma_x^2=E\{X^2\}-E^2\{X\}=\lambda$$

HW2A

2.16

P(buy) = P[X(no. of errors) ≤ 2]=P(X=0)+P(X=1)+P(X=2)

$$P(x=k)=\binom{n}{k}p^k q^{n-k}$$

$$if \quad np \approx 1 \Rightarrow \binom{n}{k}p^k q^{n-k} \approx e^{-np}\frac{(np)^k}{k!}$$

P(X=0)=0.368; P(X=1)=0.368; P(X=2)=0.184; P(X≤ 2) = 0.92

HW2A

2.23

$$f(x)=\begin{cases}\dfrac{1}{b-a}, & a \le x \le b \\ 0, & elsewhere\end{cases}$$

$$E\{X\}=\int_{-\infty}^{\infty} x f_x(x)dx=\int_{-\infty}^{\infty}\frac{x}{b-a}dx=\frac{b+a}{2}$$

$$E\{X^2\}=\int_{-\infty}^{\infty} x^2\frac{1}{b-a}dx=\frac{b^2+ab+a^2}{3}$$

$$\sigma_x^2=E\{X^2\}-E^2\{X\}=\frac{b^2+ab+a^2}{3}-\left(\frac{b+a}{2}\right)^2=\frac{(a+b)^2}{12}$$

HW2B

2.20

Let: X = no. of days of freedom

D1 = door #1 selected

D2 = door #2 selected

D3 = door #3 selected

$E\{X\} = E_D\{E_{X|D}[X|D]\} = P_{D1}E\{X|D_1\} + P_{D2}E\{X|D_2\} + P_{D3}E\{X|D_3\}$

$P_{D1}E\{X|D_1\}=0 \quad P_{D2}E\{X|D_2\}=3+E\{X\} \quad P_{D3}E\{X|D_3\}=1+E\{X\}$

E{X} = 1/3x0 + 1/3x[3+E{X}] + 1/3x [1+E{X}]

E{X} = 4 days

HW2B

2.22

$$0 \le E\{(X+\lambda Y)^2\} = E\{X^2\} + 2\lambda E\{XY\} + \lambda^2 E\{Y^2\}$$

$$let\ \lambda = -\frac{E\{XY\}}{E\{Y^2\}}, then$$

$$0 \le E\{X^2\} - \frac{E^2\{XY\}}{E\{Y^2\}} \Rightarrow E^2\{XY\} \le E\{X^2\}E\{Y^2\}$$

$$As\ 2E\{XY\} \le 2\sqrt{E\{X^2\}E\{Y^2\}}, \quad we\ have$$

$$E\{(X+Y)^2\} \le \left[\sqrt{E\{X^2\}} + \sqrt{E\{Y^2\}}\right]^2$$

$$\sqrt{E\{(X+Y)^2\}} \le \sqrt{E\{X^2\}} + \sqrt{E\{Y^2\}}$$

HW2B

2.29

$$f_{XY}(x,y)=\frac{1}{2},\quad 0\le x\le y,\quad 0\le y\le 2$$

$$f_X(x)=\int_x^2 f_{xy}(x,y)dy=\int_x^2\frac{1}{2}dy=\frac{2-x}{2},0\le x\le y$$

$$f_Y(y)=\int_0^y f_{xy}(x,y)dx=\int_0^y\frac{1}{2}dx=\frac{y}{2},0\le y\le 2$$

$$f_{X|Y}(x,y)=\frac{f_{xy}(x,y)}{f_y(y)}=\frac{1/2}{y/2}=\frac{1}{y}\;0\le y\le 2, 0\le x\le y$$

HW3A

2.26

$$\psi_x(\omega)=E\{\exp(j\omega x)\}$$

$$=E\left\{1+j\omega x+\frac{(j\omega x)^2}{2!}+\ldots+\frac{(j\omega x)^n}{n!}+\ldots\right\},|\,j\omega x|<1$$

$$=E\left\{\sum_{k=0}^{\infty}\frac{(j\omega)^k}{k!}x^k\right\}=\sum_{k=0}^{\infty}\frac{(j\omega)^k}{k!}E\{x^k\}$$

HW3A

2.28

$$\psi_x(\omega) = \int_{-\infty}^{\infty} \frac{a/\pi}{x^2 + a^2} e^{-j\omega x} dx = \int_{-\infty}^{\infty} \frac{a/\pi}{x^2 + a^2} \cos \omega x dx = e^{-a|\omega|}$$

The characteristic function of the cauchy random variable, as above, has a discontinuous first derivative at ω = 0 to give the power series of the form:

$$\psi_x(\omega) = \sum_{k=0}^{\infty} \frac{(j\omega)^k}{k!} E\{x^k\}$$

i.e. the moments of even orders are infinite

HW3A

2.33

$$X \sim N(0, \sigma_x^2), \quad Y \sim N(0, \sigma_y^2), \quad Z = \frac{1}{2}(X+Y), \quad W = \frac{1}{2}(X-Y)$$

$$X = Z + W, \quad Y = Z - W$$

$$J(Z,W) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2, \quad |J| = 2$$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right)\right\}$$

$$f_{ZW}(z,w) = \frac{1}{\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left(\frac{(z+w)^2}{\sigma_x^2} + \frac{(z-w)^2}{\sigma_y^2}\right)\right\}$$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\pi\sigma_x\sigma_y} \exp\left\{-\frac{1}{2}\left(\frac{(z+w)^2}{\sigma_x^2} + \frac{(z-w)^2}{\sigma_y^2}\right)\right\} dw$$

HW3B

2.31

$$f_{XY}(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

$$f_X(x)=\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}dy$$

$$=\frac{1}{\sqrt{2\pi}\sigma_x}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_y}{\sigma_y}\right)^2-\rho\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right]\right\}dy$$

$$=\frac{1}{\sqrt{2\pi}\sigma_x}\exp\left\{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2\right\}\times 1$$

$$f_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_y}\exp\left\{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}$$

HW3B

2.31

$$f_{X|Y}(x|y)=\frac{f_{X|Y}(x|y)}{f_Y(y)}=\frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-\frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}}{\frac{1}{\sqrt{2\pi}\sigma_y}\exp\left\{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}}$$

$$=\frac{1}{\sqrt{2\pi\sigma_x^2(1-\rho^2)}}\exp\left\{\frac{-1}{2(1-\rho^2)\sigma_x^2}\left[x-\left(\mu_x+\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right)\right]^2\right\}$$

$$E\{X\,|\,Y=y\}=\mu_x+\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y);\quad \sigma^2_{X|Y=y}=\sigma_x^2(1-\rho^2)$$

HW3B

2.43

$$\underline{X} = \begin{pmatrix} \underline{X}_1 \\ \underline{X}_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}, \quad \mu_{\underline{x}} = \begin{pmatrix} \mu_{\underline{x}1} \\ \mu_{\underline{x}2} \end{pmatrix}, \quad \Sigma_{\underline{x}} = \left(\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right)$$

$$E\{\underline{X}_1 \mid \underline{X}_2\} = \underline{\mu}_{x_1} + \Sigma_{12}\Sigma_{22}^{-1}(\underline{X}_2 - \underline{\mu}_{x_2}) = 0 + (3 \quad 2 \quad 1)\begin{pmatrix} 4 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 4 \end{pmatrix}^{-1}\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{5}{6}x_2 - \frac{1}{6}x_4$$

$E\{X_1|X_2=0.5, X_3=1.0, X_4=2.0\}=1/12$

$$E_{\underline{X}_1|\underline{X}_2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = 4 - (3 \quad 2 \quad 1)\begin{pmatrix} 4 & 3 & 2 \\ 3 & 4 & 3 \\ 2 & 3 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \frac{3}{5}$$

HW3B

2.46

Given that V_i is a normalized eigenvector

$$V_i^T V_i = 1, \quad V_i^T V_j = 0 \quad \& \quad \Sigma_x V_i = \lambda_i V_i$$

Then given $A=[V_1, \ldots, V_n]^T$ we have if

$$\underline{y} = A\underline{x} \Rightarrow \Sigma_y = A\Sigma_x A^T$$

$$= [V_1 \quad V_2 \quad \cdots \quad V_n]^T \Sigma_x [V_1 \quad V_2 \quad \cdots \quad V_n]$$

$$= [V_1 \quad V_2 \quad \cdots \quad V_n]^T [\Sigma_x V_1 \quad \Sigma_x V_2 \quad \cdots \quad \Sigma_x V_n]$$

$$= [V_1 \quad V_2 \quad \cdots \quad V_n]^T [\lambda_1 V_1 \quad \lambda_2 V_2 \quad \cdots \quad \lambda_n V_n]$$

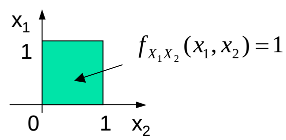
$$= \begin{bmatrix} V_1^T\lambda_1 V_1 & V_1^T\lambda_2 V_2 & \cdots & V_1^T\lambda_n V_n \\ V_2^T\lambda_1 V_1 & V_2^T\lambda_2 V_2 & \cdots & V_2^T\lambda_n V_n \\ \\ V_n^T\lambda_1 V_1 & V_n^T\lambda_2 V_2 & \cdots & V_n^T\lambda_n V_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \\ 0_1 & 0 & \cdots & \lambda_n \end{bmatrix}$$

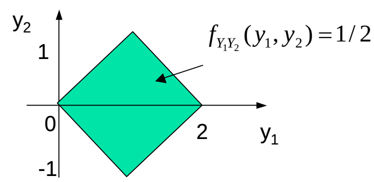
HW4A

2.38 (a)

$Y_1 = X_1 + X_2$, $Y_2 = X_1 - X_2$; $X_1 = (Y_1+Y_2)/2$, $X_2 = (Y_1-Y_2)/2$

$$f_{Y_1Y_2}(y_1,y_2) = \left.\frac{f_{X_1X_2}(x_1,x_2)}{|J(x_1,x_2)|}\right|_{x_1=(y_1+y_2)/2,\, x_2=(y_1-y_2)/2} = \left.\frac{1}{2}\right|_{0\le(y_1+y_2)\le 2,\, 0\le(y_1-y_2)\le 2}$$





HW4A

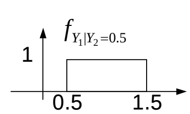
2.38 (b)

$$\rho_{y_1y_2} = \frac{E\{Y_1Y_2\} - \mu_{y_1}\mu_{y2}}{\sigma_{y_1}\sigma_{y2}} = \frac{E\{X_1^2 - X_2^2\} - E\{X_1+X_2\}E\{X_1-X_2\}}{\sigma_{y_1}\sigma_{y2}}$$

$$= \frac{E\{X_1^2 - X_2^2\} - [E^2\{X_1\} - E^2\{X_2\}]}{\sigma_{y_1}\sigma_{y2}} = 0$$

Doesn't imply independence

$$E\{Y_1 \mid Y_2 = 0.5\} = 1$$



HW4A

2.39

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{x_2} & \frac{1}{x_1} \\ \frac{}{(x_1+x_2)^2} & -\frac{}{(x_1+x_2)^2} \end{vmatrix} = -\frac{1}{x_1+x_2}$$

$$f_{Y_1Y_2}(y_1, y_2) = \begin{cases} -\frac{e^{-y_1}}{y_1}, & y_1 > 0, 0 < y_2 < 1 \\ 0, & otherwise \end{cases}$$

$$f_{Y_1}(y_1) = \int_0^1 f_{Y_1Y_2}(y_1, y_2) dy_2 = -\frac{e^{-y_1}}{y_1}$$

$$f_{Y_2}(y_2) = \int_0^\infty f_{Y_1Y_2}(y_1, y_2) dy_1 = 1$$

$f_{Y_1Y_2}(y_1, y_2) = f_{Y_1}(y_1) f_{Y_2}(y_2)$ independent

HW4A

2.40

chi – square distribution

$$Y = X^2 \sim f_Y(y) = \frac{1}{2^{\frac{1}{2}} \pi^{\frac{1}{2}}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}$$

$$\psi_Y(j\omega) = (1 - 2j\omega)^{-1/2}$$

$$Y = \sum_{i=1}^{n} X_i^2$$

$$\psi_Y(j\omega) = (1 - 2j\omega)^{-n/2} \Leftrightarrow f_Y(y) = \frac{1}{2^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} y^{\frac{n}{2}-1} e^{-\frac{y}{2}}$$

HW4B

2.35 (a)

$$f_Y(y)=\sum_{i=1}^{2}\frac{f_X(x_{(i)})}{|J(x_{(i)})|}\Bigg|_{x_{(i)}=\pm\sqrt{y}} = \sum_{i=1}^{2}\frac{\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{x_{(i)}^{2}}{2\sigma_x^2}}}{|1|}\Bigg|_{x_{(i)}=\pm\sqrt{y}} = \frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{y}{2\sigma_x^2}}$$

$$f_Y(y)=\begin{cases}\frac{1}{\sqrt{2\pi y}\sigma_x}e^{-\frac{y}{2\sigma_x^2}} & y\ge 0\\ 0 & y<0\end{cases}$$

HW4B

2.35 (b)

$$f_Y(y)=\sum_{i=1}^{2}\frac{f_X(x_{(i)})}{|J(x_{(i)})|}\Bigg|_{x_{(i)}=\pm y} = \sum_{i=1}^{2}\frac{\frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{x_{(i)}^{2}}{2\sigma_x^2}}}{|1|}\Bigg|_{x_{(i)}=\pm y} = \frac{2}{\sqrt{2\pi}\sigma_x}e^{-\frac{y^2}{2\sigma_x^2}}$$

$$f_Y(y)=\begin{cases}\frac{2}{\sqrt{2\pi}\sigma_x}e^{-\frac{y^2}{2\sigma_x^2}} & y\ge 0\\ 0 & y<0\end{cases}$$

HW4B

2.35 (c)

$$y = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$when\ y > 0,\ f_Y(y) = f_X(y)$$

$$when\ y = 0,\ P\{y = 0\} = P\{x < 0\} = \frac{1}{2} \Rightarrow f_Y(y) = \frac{1}{2}\delta(y)$$

$$f_Y(y) = \begin{cases} \frac{1}{2}\delta(y) + \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{y^2}{2\sigma_x^2}} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

HW4B

2.35 (d)

$$y = \begin{cases} 1 & x > \sigma_x \\ x & |x| \leq \sigma_x \\ -1 & x < -\sigma_x \end{cases}$$

$$P(y = 1) = P(x > \sigma_x) = Q(1)$$

$$P(y = 1) = P(x < -\sigma_x) = Q(1)$$

$$f_Y(y) = \begin{cases} Q(1) & y = -1 \\ \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{y^2}{2\sigma_x^2}} & |y| < \sigma_x \\ Q(1) & y = 1 \\ 1 & elsewhere \end{cases}$$

HW4B

2.37

$$X_1 = r\cos\theta, \quad X_2 = r\sin\theta$$

$$\left|J(x_1,x_2)\right|_{x_1=r\cos\theta,x_2=r\sin\theta} = \left|J(r,\theta)\right|^{-1} = \begin{vmatrix}\cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta\end{vmatrix}^{-1} = \frac{1}{r}$$

$$f_{R\Theta}(r,\theta) = \left.\frac{f_{X_1X_2}(x_1,x_2)}{\frac{1}{r}}\right|_{x_1=r\cos\theta,x_2=r\sin\theta} = rf_{X_1X_2}(r\cos\theta, r\sin\theta) = \frac{r}{2\pi\sigma_x^2}\exp\left\{-\frac{r^2\cos^2\theta}{2\sigma_x^2} - \frac{r^2\sin^2\theta}{2\sigma_x^2}\right\}$$

$$= \frac{r}{2\pi\sigma_x^2}\exp\left\{-\frac{r^2}{2\sigma_x^2}\right\}, \textit{ where } r \ge 0, 0 \le \theta < 2\pi$$

$$f_\Theta(\theta) = \frac{1}{2\pi}\int_0^\infty \frac{r}{\sigma_x^2}\exp\left\{\frac{-r^2}{2\sigma_x^2}\right\}dr = \frac{1}{2\pi}, 0 \le \theta < 2\pi$$

$$f_R(r) = \int_0^{2\pi}\frac{r}{2\pi\sigma_x^2}\exp\left\{\frac{-r^2}{2\sigma_x^2}\right\}d\theta = \frac{r}{\sigma_x^2}\exp\left(\frac{-r^2}{2\sigma_x^2}\right), \quad r \ge 0$$

$$f_{R\Theta}(r,\theta) = f_\Theta(\theta)f_R(r)$$ independent

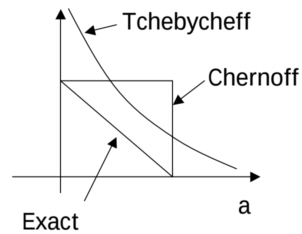
HW5A

2.48

$$E\{U(X)\} = P[U(X) \ge a]E\{U(X) \mid U(X) \ge a\} + P[U(X) < a]E\{U(X) \mid U(X) < a\}$$
$$\ge P[U(X) \ge a]E\{U(X) \mid U(X) \ge a\}$$
$$\ge P[U(X) \ge a]a$$

HW5A

2.49 (a)



$X \sim U(0,1)$

$$Exact: P(X \geq a) = \int_a^1 1dx = \begin{cases} 1-a & 0 \leq a \leq 1 \\ 0 & elsewhere \end{cases}$$

$$Tchebycheff: P(|X| \geq a) \leq \frac{1}{a^2} E\{X^2\} = \frac{1}{a^2}\int_0^1 x^2 dx = \frac{1}{3a^2}$$

$$Chernoff: P(X > a) \leq \min e^{-at} E\{e^{tx}\}$$

$$E\{e^{tx}\} = \int_0^\infty e^{tx} dx = \frac{1}{t}(e^t - 1)$$

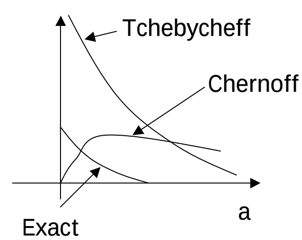
$$let\ g(t) = \frac{e^{-at}}{t}(e^t - 1),\ then\ g'(t) = -\frac{1}{t}e^t + \frac{1}{t} - ae^t + a = (1+at)(1-e^t) = 0$$

$$\Rightarrow t = 0 \Rightarrow \min_t g(t) = \lim_{t \to 0} g(t) = 1$$

$$P(X > a) \leq 1,\ for\ 0 < a < 1$$

HW5A

2.49 (b)



$f_X(x) \sim e^{-x} \quad x > 0$

$$Exact: P(X \geq a) = \int_a^\infty e^{-x} dx = e^{-a} \quad a > 0$$

$$Tchebycheff: P(|X| \geq a) \leq \frac{1}{a^2} E\{X^2\} = \frac{1}{a^2}\int_0^\infty x^2 e^{-x} dx = \frac{2}{a^2}$$

$$Chernoff: P(X > a) \leq \min e^{-at} E\{e^{tx}\}$$

$$E\{e^{tx}\} = \int_0^\infty e^{tx} e^{-x} dx = \frac{1}{1-t}$$

$$let\ g(t) = \frac{e^{-at}}{1-t},\ then\ g'(t) = 0 \Rightarrow (1-t)(-a) + 1 = 0 \Rightarrow t = \frac{a-1}{a}$$

$$P(X > a) \leq ae^{1-a},\ for\ a > 0$$

HW5A

2.49 (C)

$X \sim N(0,1)$

$Exact: P(X \geq a) = Q(a)$

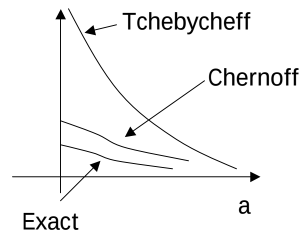
$Tchebycheff: P(|X| \geq a) \leq \frac{1}{a^2} E\{X^2\} = \frac{1}{a^2}$

$Chernoff: P(X > a) \leq \min e^{-at} E\{e^{tx}\}$

$$E\{e^{tx}\} = \int_0^\infty e^{tx} N_x(0,1) dx = e^{\frac{t^2}{2}}$$

$$let\ g(t) = e^{\left(\frac{t^2}{2} - at\right)},\ then\ g'(t) = 0 \Rightarrow t = a$$

$$P(X > a) \leq e^{-\frac{a^2}{2}}$$



HW5A

2.59

$$f_X(x) = \frac{a/\pi}{x^2 + a^2}, \quad \psi_X(\omega) = e^{-a|\omega|}$$

$$\psi_{X_1 + \ldots + X_n}(\omega) = e^{-a|\omega|} \ldots e^{-a|\omega|} = e^{-na|\omega|}$$

$$\psi_Y(\omega) = e^{-a|\omega|}, \quad f_Y(y) = \frac{a/\pi}{y^2 + a^2}$$

Y does not tend to a normal density, as

$$\sigma_x^2 = E\{X\} = \int_{-\infty}^{\infty} \frac{x^2 a}{\pi(x^2 + a^2)} dx = \infty$$

HW5B

2.31

HW3B

HW5B

2.46

HW3B

HW5B

2.55

$$E\{\overline{X}\} = E\left\{\frac{1}{n}\sum_{i=1}^{n} x_i\right\} = \frac{1}{n}\sum_{i=1}^{n} E\{x_i\} = 0$$

$$Var\{\overline{X}\} = E\{\overline{X}^2\} - E^2\{\overline{X}\} = E\left\{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j\right\}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E\{x_i x_j\} = \frac{1}{n^2}\sum_{i=1}^{n} \sigma_x^2 = \frac{\sigma_x^2}{n}$$

$$\lim_{n\to\infty} E\left\{(\overline{X} - 0)^2\right\} = \lim_{n\to\infty} \frac{\sigma_x^2}{n} = 0$$

$$l.i.m\overline{X} = 0$$