Bivariate Gaussian

Lecture 8

EE 640
Stochastic Systems

Outline

- Linearity of Expectation
- Correlation Coefficient
- Joint (Bivariate)Gaussian pdf
- Conditional Statistics

Linearity of Expectation

$$E\{aX + bY\} = aE\{X\} + bE\{Y\}$$

$$E\left\{\sum_{i=1}^{N} a_i X_i\right\} = \sum_{i=1}^{N} \alpha_i E\{X_i\}$$

Conditional Expectation

$$E\{E\{g(X,Y) \mid X\}\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(x,y) f_{Y|X}(y \mid x)dy\right] f_X(x)dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{Y|X}(y \mid x) f_X(x)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{XY}(x,y)dxdy$$

Correlation Coefficient

$$\rho_{xy} = \frac{\sigma_{xy}}{\sqrt{\sigma_x^2 \sigma_y^2}} = \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

$$\mu_x = E\{X\}, \quad \mu_y = E\{Y\}$$

$$\sigma_x^2 = E\{(X - \mu_x)^2\}$$

$$\sigma_y^2 = E\{(X - \mu_x)^2\}$$

$$\sigma_{xy} = E\{(X - \mu_x)(Y - \mu_y)\}$$

Uncorrelated $E\{XY\} = E\{X\}E\{Y\}$

Orthogonal $E\{XY\} = 0$

If X independent of Y, then uncorrelated
If uncorrelated and $\mu_x = \mu_y = 0$, then orthogonal

If $\rho = 0$, then uncorrelated
If $\rho = 1$, then $\sigma_{xy} = \sigma_x \sigma_y = \sigma^2$

Joint Gaussian pdf (Bivariate Guassian)

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}$$

$$\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

$\rho = 0$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left\{-\frac{1}{2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x}\exp\left\{-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right\}\frac{1}{\sqrt{2\pi}\sigma_y}\exp\left\{-\frac{(x-\mu_y)^2}{2\sigma_y^2}\right\}$$

$$= f_X(x) f_Y(y)$$

Example 2.33

$$X \sim N(0,\sigma_x^2) \quad Y \sim N(0,\sigma_y^2)$$

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

$$Z = \frac{1}{2}(X+Y) \quad W = \frac{1}{2}(X-Y)$$

(1)f_{ZW} (z, w) = ? (2) find marginals (3) Are Z and W independent?

Sum or difference of Gussians is another Gaussian

Example Cont'

(2) E{Z} = ½ (E{X} + E{Y}) = 0 E{W} = ½ (E{X} + E{Y}) = 0

σ_z^2 = E{Z^2} = ¼ E{X^2 + 2XY + Y^2}
= ¼ (E{X^2} + 2E{XY} + E{Y^2}
= [σ_x^2 + σ_y^2]/4

σ_W^2 = E{Z^2} = ¼ E{X^2 - 2XY + Y^2}
= ¼ (E{X^2} - 2E{XY} + E{Y^2}
= [σ_x^2 + σ_y^2]/4

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_z} e^{-\frac{z^2}{2\sigma_z^2}} \qquad f_W(z) = \frac{1}{\sqrt{2\pi}\sigma_W} e^{-\frac{w^2}{2\sigma_W^2}}$$

Example Cont'

(3) E{ZW} = ¼ E{X^2 – Y^2} = [σ_x^2 – σ_y^2]/4

$$\rho_{zw} = \frac{\sigma_{zw}}{\sigma_z \sigma_w} = \frac{\frac{\sigma_x^2 - \sigma_y^2}{4}}{\frac{\sigma_x^2 + \sigma_y^2}{4}} = \frac{\sigma_x^2 - \sigma_y^2}{\sigma_x^2 + \sigma_y^2}$$

If σ_x^2 = σ_y^2, then Z and W are independent, else not independent

Example Cont'

(3) $$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)\sigma^2}\left[z^2 - 2\rho zw + w^2\right]\right\}$$

Where $\sigma^2 = \sigma_z^2 = \sigma_w^2$, $\rho = \rho_{zw}^2$

Conditional Statistics

$$F_X(x) = \sum_{i=1}^{n} F_X(x \mid A_i) P(A_i)$$

$$\frac{\partial F_X(x)}{\partial x} = f_X(x) = \sum_{i=1}^{n} \frac{\partial F(x \mid A_i)}{\partial x} P(A_i)$$

Where $\frac{\partial F(x \mid A_i)}{\partial x} = \frac{\partial}{\partial x} \int_{-\infty}^{x} f(\lambda \mid A_i) d\lambda$

$$f_X(x) = \sum_{i=1}^{n} f_X(x \mid A_i) P(A_i)$$