Random Vectors (RVs)

Lecture 7

EE 640
Stochastic Systems

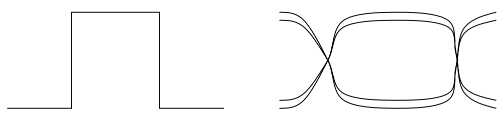
Outline

- Applications
- Definition
- Covariance Matrix
- Correlation matrix

Random Vectors (RV)

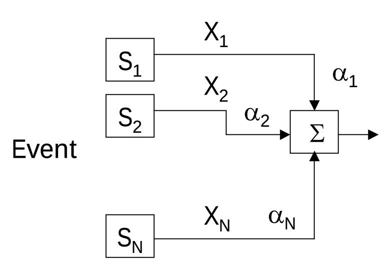
RVs are used to represent multiple r.v.s. Two common uses of RVs are
(1) Represent time samples of random signals/processes and
(2) represent processed parametric values from different processing unite, such as sensors or filter branches.

EX: Signals. Assume a binary signal with less than perfect synchronization



Eye diagram

Example: Sensor Fusion



$$Z = \sum_{n=1}^{N} X_n \alpha_n$$

We can optimize α_n values to yield high SNR or high discrimination between different events

RV Definition

Let, $\underline{X} = \left[\underline{X_1} \quad \underline{X_2} \quad \cdots \quad \underline{X_N}\right]^T$ NX1 column vector.

$$F_{\underline{X}}(\underline{x}) = P\{X_1 < x_1, X_2 < x_2, \cdots, X_N < x_N\}$$
$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f_{\underline{X}}(\underline{x}) dx_1 \ldots dx_N$$

$$E\{\underline{X}\} = \left[E\{X_1\} \quad E\{X_2\} \quad \cdots \quad E\{X_N\}\right]^T = \underline{\mu_x}$$

where $\quad E\{X_n\} = \mu_n \quad \mathrm{var}\{X_n\} = \sigma_n^2$

Covariance matrix

$$\underline{\Sigma}_x = E\left\{(\underline{X} - \underline{\mu}_x)(\underline{X} - \underline{\mu}_x)^T\right\}$$

NXN matrix

$$= E\left\{\underline{X}\,\underline{X}^T\right\} - E\left\{\underline{\mu}_x \underline{X}^T\right\} - E\left\{\underline{X}\,\underline{\mu}_x^T\right\} + E\left\{\underline{\mu}_x \underline{\mu}_x^T\right\}$$
$$= E\left\{\underline{X}\,\underline{X}^T\right\} - \underline{\mu}_x \underline{\mu}_x^T$$
$$= \begin{bmatrix} E\{X_1 X_1^T\} & E\{X_1 X_2^T\} & \cdots & E\{X_1 X_N^T\} \\ \vdots & & & \vdots \\ E\{X_N X_1^T\} & & & E\{X_N X_N^T\} \end{bmatrix} - \begin{bmatrix} \mu_1^2 & \mu_1\mu_2 & \cdots & \mu_1\mu_N \\ \vdots & & & \vdots \\ \mu_N\mu_1 & \mu_N\mu_2 & \cdots & \mu_N^2 \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2 & \cdots & \sigma_1\sigma_N \\ \vdots & & & \vdots \\ \sigma_N\sigma_1 & \sigma_N\sigma_2 & \cdots & \sigma_N^2 \end{bmatrix}$$

Note Σ_x is symmetric and main diagonal Contains marginal variances.

Gaussian Random Vector

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{N} f_{X_i}(x_i)$$

Multivariant Guassian pdf

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N} \left|\Sigma_x\right|^{1/2}} \exp\left\{ -\frac{1}{2} (\underline{X} - \underline{\mu}_x)^T \Sigma_x^{-1} (\underline{X} - \underline{\mu}_x) \right\}$$

Determinant

1xN NxN Nx1

Example I

$$E\{X_1\} = 6 = \mu_1$$
$$E\{X_2\} = 0 = \mu_2$$
$$E\{X_3\} = 0 = \mu_3$$

$$E\{\underline{X}\} = \begin{bmatrix} 6 \\ 0 \\ 8 \end{bmatrix} = \underline{\mu}_x$$

$$Y_1 = X_1 - X_2 \qquad E\{Y_1\} = \mu_1 - \mu_2$$
$$Y_2 = X_1 + X_2 - 2X_3 \qquad E\{Y_2\} = \mu_1 + \mu_2 - 2\mu_3$$
$$Y_1 = X_1 + X_2 \qquad E\{Y_1\} = \mu_1 + \mu_2$$

$$\underline{\mu}_y = E\left\{ \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\} = \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 + \mu_2 - 2\mu_3 \\ \mu_1 + \mu_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -10 \\ 6 \end{bmatrix}$$

Example II

$$E\{\underline{Y}\underline{Y}^T\} = E\left\{\begin{bmatrix} Y_1^2 & Y_1Y_2 & Y_1Y_3 \\ Y_2Y_1 & Y_2^{\,2} & Y_2Y_3 \\ Y_3Y_1 & Y_3Y_2 & Y_3^2 \end{bmatrix}\right\}$$

If y_i independent of y_j for $i \neq j$

$$= \begin{bmatrix} \sigma_1^2 + \mu_1^2 & \mu_1\mu_2 & \mu_1\mu_3 \\ \mu_2\mu_1 & \sigma_2^2 + \mu_2^2 & \mu_2\mu_3 \\ \mu_3\mu_1 & \mu_3\mu_2 & \sigma_3^2 + \mu_3^2 \end{bmatrix}$$

$$\Sigma_y = E\{\underline{Y}\underline{Y}^T\} - \underline{\mu}_y \underline{\mu}_y^T = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$$

Correlation Matrix

$$R_{xx} = E\{\underline{X}\,\underline{X}^T\} \qquad \underline{X} = [X_1 \quad X_2 \quad \cdots \quad X_N]^T$$

$$C_{xx} = E\{\underline{X}\,\underline{X}^T\} - \underline{\mu}\underline{\mu}_x^T = R_{xx} - \underline{\mu}\underline{\mu}_x^T$$

R_{xx} is non-negative definite

$Q = \underline{a}^T R_{xx} a \geq 0$ for $\forall\ \underline{a}$

Iff $Q > 0$, then R_{xx} is positive definite and X_i are linearly independent

Iff $Q = 0$, then X_i are linearly dependent

$$\Delta_{xx} = \det\{R_{xx}\}$$

Iff $\Delta > 0$, then X_i are linearly independent

Iff $\Delta = 0$, then X_i are linearly dependent