Characteristic Function

Lecture 6

EE 640
Stochastic Systems

Outline

- Characteristic Function
- Moment Generating Functions

Characteristic Function

$$\Phi_X(\omega) \stackrel{\Delta}{=} E\left\{e^{j\omega X}\right\} = \int_{-\infty}^{\infty} f_X(x) e^{j\omega x} dx$$

For independent r.v.s sum $Z = X_1 + X_2 + \ldots + X_N$

the pdf is the convolution of pdfs $f_Z(z) = f_{X1}(z) * f_{X2}(z) * \ldots * f_{XN}(z)$

The properties of Fourier transforms are such that

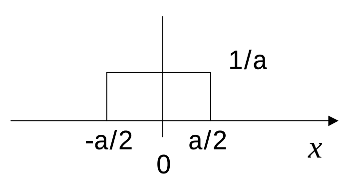
$$\Phi_Z(\omega) = \prod_{i=1}^{N} \Phi_{Xi}(\omega)$$

In fact, for $Z = \Sigma_i X_i$, the above equation is true,
then X_i are independent

Example I

X and Y are independent identically distributed (iid)

With $f_X(x) = 1/a \, rect(x/a)$.



Let Z = X + Y

$$\Phi_Z(\omega) = \Phi_X(\omega)\Phi_Y(\omega)$$

$$\Phi_X(\omega) = \Phi_Y(\omega) = \int_{-\infty}^{\infty} \frac{1}{a} rect\left(\frac{\lambda}{a}\right) e^{j\omega\lambda} d\lambda = \frac{1}{a} \int_{-a/2}^{a/2} e^{j\omega\lambda} d\lambda$$

$$= \frac{1}{a} \frac{e^{j\omega\lambda}}{j\omega}\bigg|_{-a/2}^{a/2} = \frac{1}{j\omega a}\left(e^{j\omega a/2} - e^{-j\omega a/2}\right)$$

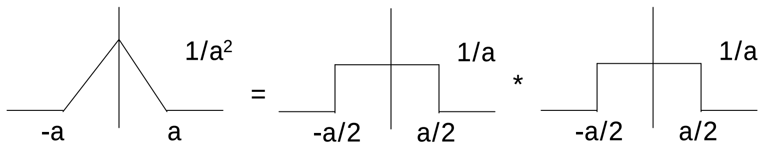
Example I Cont’

$$= \frac{1}{\omega a/2}\frac{e^{j\omega a/2} - e^{-j\omega a/2}}{2j} = \frac{\sin\left(\frac{\omega a}{2}\right)}{\frac{\omega a}{2}} = \sin c\left(\frac{\omega a}{2\pi}\right) = Sa\left(\frac{\omega a}{2}\right)$$

Couches Sinc

$$\Phi_Z(\omega) = Sa^2\left(\frac{\omega a}{2}\right)$$

$f_Z(z)$ =



Example II

Find $\Phi(\omega)$ of

$$f_X(x) = \frac{1}{\pi[1+(x-\mu)^2]}$$

$y = x - \mu$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} \frac{1}{\pi[1+(x-\mu)^2]} e^{j\omega x} dx = e^{j\omega\mu} \int_{-\infty}^{\infty} \frac{1}{\pi[1+y^2]} e^{j\omega y} dy$$

$$= \frac{e^{j\omega\mu}}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+y^2} e^{j\omega y} dy = e^{j\omega\mu} e^{-|\omega|} = e^{j\omega\mu - |\omega|}$$

$$e^{-|\omega|} \overset{F}{\leftrightarrow} \frac{1}{2\pi}\frac{2}{1+y^2}$$

Example III

Gaussian r.v.s

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \qquad \Phi_X(\omega) = \exp\left[j\mu\omega + \frac{(\sigma j\omega)^2}{2}\right]$$

X and Y are independent X ~ N(μ_x, σ_x^2), Y ~ N(μ_y, σ_y^2)

Z = aX + bY

$$\Phi_Z(\omega) = E\{e^{j\omega Z}\} = E\{e^{j\omega aX + j\omega bY}\} = E\{e^{j\omega aX}\}E\{e^{j\omega bY}\}$$
$$= \exp\{j\omega(a\mu_x + b\mu_y) + (a^2\sigma_x^2 + b^2\sigma_y^2)(j\omega)^2/2\}$$

$$\mu_z = a\mu_x + b\mu_y$$
$$\sigma_z^2 = a^2\sigma_x^2 + b^2\sigma_y^2$$

Moment Generating Functions

For a r.v. X, $\theta(t) \stackrel{\Delta}{=} E\{e^{tX}\}$

where *t* is a complex variable.

$$\theta(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$ Similar to a bilinear Laplace transform.

We can expend $e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + \ldots + \frac{(tX)^n}{n!} + \ldots$

$$E\{e^{tX}\} = 1 + t\mu + \frac{t^2\delta_2}{2!} + \ldots + \frac{t^n\delta_n}{n!} + \ldots$$ where δ_n = E{X^n}.

$$\delta_k = \theta^{(k)}(0) \stackrel{\Delta}{=} \left.\frac{d^k\theta(t)}{dt^k}\right|_{t=0}$$

Example

Gaussian r.v. X ~ N(μ, σ²)

$$\theta(t) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{tx} dx = e^{\left(\mu t + \frac{t^2\sigma^2}{2}\right)} \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{[x-(\mu+t\sigma^2)]^2}{2\sigma^2}} dx$$

$$= e^{\left(\mu t + \frac{t^2\sigma^2}{2}\right)}$$

$$\theta^0(t)\Big|_{t=0} = 1 \qquad \theta^1(t)\Big|_{t=0} = \frac{d\theta(t)}{dt}\Big|_{t=0} = (\mu + \sigma^2 t)e^{(\mu t + \sigma^2 t^2/2)}\Big|_{t=0} = \mu$$

Example Cont'

$$\beta = \mu t + \frac{\sigma^2 t^2}{2}, \quad \frac{d\beta}{dt} = \mu + \sigma^2 t$$

$$\theta^2(t)\Big|_{t=0} = \frac{d(\mu+\sigma^2 t)}{dt} e^{\beta} + (\mu+\sigma^2 t)\frac{d\beta}{dt} e^{\beta}\Big|_{t=0}$$

$$= \sigma^2 e^{\beta} + (\mu^2 + 2\mu\sigma^2 t + \sigma^4 t^2)e^{\beta}\Big|_{t=0} = \sigma^2 + \mu^2 = E\{X^2\}$$

$$\theta^3(t)\Big|_{t=0} = \sigma^2(\mu+\sigma^2 t)e^{\beta} + (2\mu\sigma^2 + \sigma^4 t^2)e^{\beta} + (\mu^2 + 2\mu\sigma^2 t + \sigma^4 t^2)(\mu+\sigma^2 t)e^{\beta}\Big|_{t=0}$$

$$= \sigma^2\mu + 2\mu\sigma^2 + \mu^3 = 3\mu\sigma^2 + \mu^3$$

Comments

This result is consistent with the basic form of the moment equation

$$E\{X^n\} = \int_{-\infty}^{\infty} x^n f_X(x)dx$$

Central moments are defined as

$$E\{(X-\mu)^n\} = \int_{-\infty}^{\infty} (x-\mu)^n f_X(x)dx$$