Expected Values

Lecture 5

EE 640
Stochastic Systems

Outline

- Mean
- Variance
- The Dirac Delta Function
- Conditional and Joint Distribution

Definition

We can numerically approximate mean and variance for a finite number of sample values as for N samples x_i

$$mean \equiv \mu_S = \frac{1}{N}\sum_{i=1}^{N} x_i$$

Estimated mean value

$$\text{variance} \equiv \sigma_S^{\ 2} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \mu_S)^2$$

Unbiased variance

Using Probability Density functions

IF we know the statistics (I.e. pdf or cdf) of a r.v., we can find the mean and variance exactly:

X ~ $f_X(x)$and y = g(x)

Ensemble Mean

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x)dx$$

$$\mu = E\{X\} = \int_{-\infty}^{\infty} x f_X(x)dx$$

$$\sigma^2 = E\{(X-\mu)^2\} = \int_{-\infty}^{\infty}(x-\mu)^2 f_X(x)dx$$

$$= \int_{-\infty}^{\infty} x^2 f_X(x)dx - \int_{-\infty}^{\infty} 2\mu x f_X(x)dx + \int_{-\infty}^{\infty} \mu^2 f_X(x)dx$$

Second Moment

$$= E\{X^2\} - 2\mu^2 + \mu^2 = E\{X^2\} - \mu^2$$

Example

Given a uniform r.v. X ~ U(0, 1)

$$\mu = E\{X\} = \int_{-\infty}^{\infty} x f_X(x) dx = \int_0^1 x dx = \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{2}$$

$$E\{X^2\} = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

$$\sigma^2 = E\{X^2\} - \mu^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

Using Probability Mass Function

Consider a discrete valued r. v. *X*,

Let p_i = Prb(X=x_i) for *i* = 1, 2, ... N

Then 1= Σp_i, where Prb(X=x_i) is a probability mass function

The mean value and variance are

$$E\{X\} = \sum_{i=1}^{N} x_i p_i$$

If p_i = p, then $E\{X\} = \sum_{i=1}^{N} x_i \frac{1}{N} = \frac{1}{N}\sum_{i=1}^{N} x_i$

$$E\{(X-\mu)^2\} = \sum_{i=1}^{N} (x_i - \mu)^2 p_i$$

If p_i = p, then

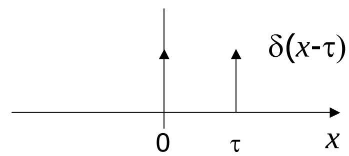
$$E\{(X-\mu)^2\} = \sum_{i=1}^{N} (x_i - \mu)^2 \frac{1}{N} = \frac{1}{N}\sum_{i=1}^{N} (x_i - \mu)^2$$

Dirac Delta Function

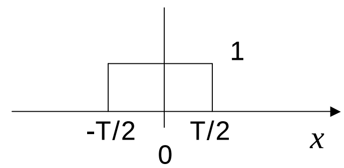
$\delta(x)$ is defined as $\delta(x) = 0$, for $\forall x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x)dx = 1$

It can be approximated as

$$\delta(x) = \lim_{a \to \infty} a \, rect\left(\frac{x}{1/a}\right)$$



$$rect\left(\frac{x}{T}\right) = \begin{cases} 1 & for \, |x| \leq \frac{T}{2} \\ 0 & elsewhere \end{cases}$$



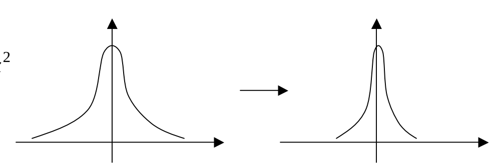
Properties of Delta Function

$$U(x) = \int_{-\infty}^{\infty} \delta(x)dx = \begin{cases} 1 & x > 0 \\ 0 & else \end{cases}$$

$$h(x) = h(x) * \delta(x) = \int_{-\infty}^{\infty} \delta(x) h(x - \lambda) dx$$

It can also be approximated as

$$\delta(x) = \lim_{a \to \infty} a \, e^{-\pi a^2 x^2}$$



Conditional Distribution

Given a r. v. *X* that is conditional on event B we have the conditional cdf as

$F_X(x|B) = P(X \le x, B)/P(B)$

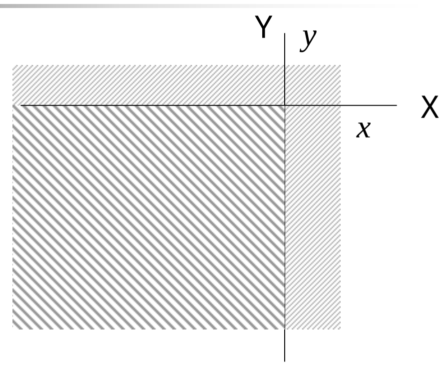
The conditional pdf is then

$f_X(x|B) = d\, F_X(x|B)\, /dx$

$F_{XY}(x, y) = P(X \le x, Y < y)$

Joint Distribution

$F_{XY}(x, y) = P(X \le x, Y \le y)$



The joint pdf is

$$f_{XX}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\lambda, \beta) d\lambda d\beta$$

Properties of Joint Distribution

$P(y_1 < y < y_2) = F_{XY}(\infty, y_2) - F_{XY}(\infty, y_1)$

For independent r. v.

$F_{XY}(x, y) = F_X(x)\, F_Y(x)$, $f_{XY}(x, y) = f_X(x)\, f_Y(x)$

Marginal

$F_X(x) = F_{XY}(x, \infty)$, $F_Y(y) = F_{XY}(\infty, y)$

Gaussian joint pdf

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} e^{-\left(\frac{x^2+y^2}{2\sigma^2}\right)}$$