Random Variables

Lecture 4

EE 640
Stochastic Systems

Outline

- Random Variables
- Probability Measurement of Random Variables
  - Probability Distribution Function
  - Probability Density Function
- Gaussian Random Variables
- Uniform Random Variables
- Random Variables with Exponential Distributions

Definition

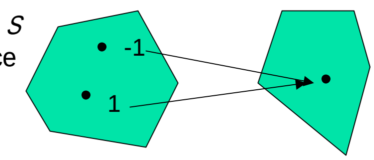
A random variable X is a real function of elements of a sample space S

[Ex] $X(s)$ is a random variable as a function of S

$X = X(s) = s^2$ Let $0 < s < 12$, then $0 < x < 144$

Sampling 2:1 mapping or non-unique mapping

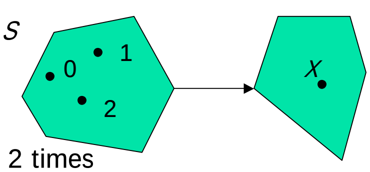
Sample space S
Outcome space



Ensemble of the random variable $S_X \subset$
Real number space R

An Example of the Random Variable

Sample space S



Tossing a coin 2 times
3 outcomes

Described by an experiment

Ensemble of the random variable S_X

Random variable X: number of head appearance

Described by real numbers

Event: “head appear once”

Described by words

Probability Measure of Random Variables

Like Random Events there are axioms to define a random variable

$P\{X \le x\} \equiv$ the probability that the random variable X is less than or equal to a deterministic value.

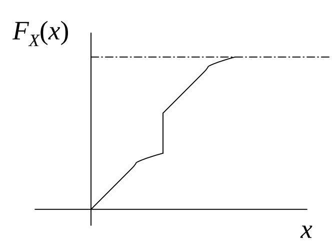
$P\{X = \infty\} = P\{X = 0\} = 0$

We already defined a probability measure so

$0 \le P\{X = \infty\} \le 1$

Discrete and Continuous Random Variables (R.V.)

- Discrete R.V. have only discrete values
  - Binomial R.V., Poisson R.V.
  - Multi-nomial R.V.
  - Probability mass function
- Continuous R.V. have continuous values
  - Gaussian R.V.
  - Gamma R.V.
  - Probability density function
- They can be mixed



Cumulative Probability Distribution Function

$F_X(x) = P\{X \leq x\}$

Important Properties

(1) $F_X(-\infty) = 0$; $F_X(\infty) = 1$

(2) $0 \leq F_X(x) \leq 1$, $F_X(x_1) \leq F_X(x_2)$, if $x_1 \leq x_2$

(3) $F_X(x+) = F_X(x)$

(4) $P(x_1 < X < x_2) = F_X(x_2) - F_X(x_1)$

Probability Density Function

$f_X(x) = dF_X(x)/dt$

In a less rigorous way the pdf represents the probability of $X = x$, For continuous r.v.s this is only a conceptual aid since the probabilities of a continuous r.v. having a specific value: $P(X = x) = 0$

Properties of a pdf:

(1) $0 \leq f_X(x), \forall x$

(2) $\int_{-\infty}^{\infty} f_X(x)dx = 1$

(3) $F_X(x) = \int_{-\infty}^{x} f_X(t)dt$

(4) $P\{x_1 \leq X \leq x_1\} = \int_{x_1}^{x_2} f_X(x)dx$

The Gaussian Random Variable

pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}$$

X ~ $N(\mu, \sigma^2)$

cdf

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} dx$$

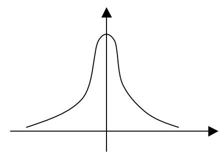
μ_x = mean value
σ_x^2 = variance
σ_x = standard deviation

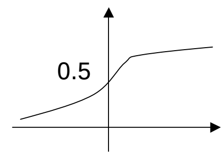
"$F_X(x)$" values can be obtained from tables but these values are for a "normalized" value of x.

Normalized Gaussian Random Variable

The normalized case is where $\mu = 0$ and $\sigma^2 = 1$

For negative x values we have $F(-x) = 1 - F(x)$. Note that $F(0) = 0.5$ for the normalized case.





We make a variable change $\lambda = (x - \mu)/\sigma$

$F_X(x) = F[(x - \mu)/\sigma]$ or $F_X(x) = F(\lambda)$

An Example

Given a Gaussian r.v. with μ = 3 and σ =2

Find P(X ≤ 5.5), x = 5.5, λ = (5.5 – 3)/2 = 1.25

F(λ) = 0.5987

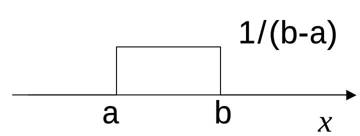
The Q function

$$Q(x) = 1 - F(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{\beta^2}{2}} d\beta$$

Uniform Random Variables

X ~ U(a, b) ≡ X is a r. v. having a uniform distribution between a and b. The pdf is

$$U(x) = \begin{cases} \frac{1}{b-a} & for\ a < x < b \\ 0 & elsewhere \end{cases}$$



$$\int_{-\infty}^{\infty} U(x) dx = \int_a^b \frac{1}{b-a} dx = \frac{1}{b-a} x \Big|_a^b = 1$$

An Example

Model coin toss with uniform r. v.

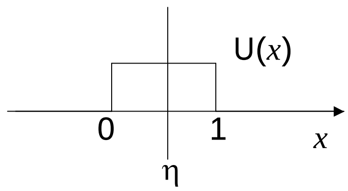
For a coin heads has P(H) = p and tails P(T) = q.

The model requires a threshold η and a uniformly distributed random number generators.

Let X ~ U(0, 1) be approximated by the MATLAB pseudo random number generator RAND( );

If X < η, then call tails

if X >= η, then call heads



An Example Cont'

$$P(tails) = F_X(\eta) = \int_{-\infty}^{\eta} u(x)dx = \int_{0}^{\eta} 1dx = \eta$$

$$P(heads) = 1 - P(tails) = \int_{\eta}^{\infty} u(x)dx = \int_{\eta}^{1} 1dx = 1 - \eta$$

where 0<=η<=1. For a fair coin, η=0.5

Exponential Distribution

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad F_X(x) = \begin{cases} \int_{-\infty}^{x} f_X(x)dx = 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P\{X > t+s \mid t > s\} = \frac{P\{X > t+s, X > s\}}{P\{X > s\}} == \frac{P\{X > t+s\}}{P\{X > s\}} \frac{1 - F_X(t+s)}{F_X(s)}$$

$$= \frac{e^{-(t+s)}}{e^{-s}} = e^{-t} = P\{X > t\}$$

As long as a device is "alive" at *s*, the prob. of life time is only dependent on *t*, which means, no memory. (Memoryless property)

Poisson Probability Mass Function

$$f_K(k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

$$\sum_{k=0}^{\infty} f_K(k) = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} = 1$$

Gamma Distribution

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{\frac{x}{\beta}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

α and β are parameters and result in a variety of pdf shapes

where

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

When α=n, Γ(n)=(n-1) Γ(n-1)=(n-1)!, β = 1/λ

$$f_X(x) = \begin{cases} \dfrac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Rayleigh Distribution

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\frac{x^2}{2\sigma^2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Chi-Sqaure Distribution

$$f_X(x) = \begin{cases} \dfrac{x^{n/2-1}}{2^{n/2}\Gamma(n/2)} e^{-\frac{x}{2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

X is chi-square $X^2(n)$ with n degree of freedom