Set & Probability

Lecture 2

EE 640
Stochastic Systems

Set Definition

A, B : represents Sets
a, b : represents elements/members of sets
ϕ : empty or Null set
S : whole/universe set

Countable : A set is "countable" if its elements can be indexed by integers

Finite : A countable set is "finite" if its elements has a finite number of elements

Subset

$A \subset B$: A is contained by B, A is a subset of B

$\supset$

$\phi \subset A,\ A \subset S,\ A \subset A$

$A = B$: A is equal to B iff $A \subset B$ and $A \supset B$

$N \subset Z \subset R$

Set Operation

Union:

$A \cup B \quad A+B \quad A_1 \cup A_2 \cup \ldots \cup A_N = \bigcup_{i=1}^{N} A_i$

Intersection:

$A \cap B \quad AB \quad A_1 \cap A_2 \cap \ldots \cap A_N = \bigcap_{i=1}^{N} A_i$

Complement: $\overline{A}$ $\quad \overline{A} \cup A = S$

$\overline{A} \cap A = \phi$

Operator precedence: (*) >– > $\cap$ > $\cup$

$\cup$ and $\cap$ are commutative, associative, and distributive

DeMorgan's Law

$\overline{(A \cup B)} = \bar{A} \cap \bar{B}$

$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$

Mutually exclusive:

$A \cap B = \phi$

Probability of Events

Axioms

(1) $P(S) = 1$

(2) $P(A) \geq 0$, for all $A \subset S$

(3) If $A \cap B = \phi$, $P(A \cup B) = P(A) + P(B)$

Properties

(1) $P(\phi)=0$: $A \cap \phi=\phi$ $A \cup \phi=A$

$P(A \cup \phi) = P(A) + P(\phi) = P(A)$

(2) $1 \geq P(A)$

(3) $P(\bar{A}) = 1 - P(A)$: $1= P(S) = P(\bar{A} \cup A) = P(\bar{A}) + P(A)$

(4) If $A \subset B$, then $P(A) \leq P(B)$

(5) $P(A \cup B) = P(A) + P(B) - P(AB)$

$A \cup B = A \cup \bar{A}B$, $P(A \cup B) = P(A) + P(\bar{A}B)$

$B = AB \cup \bar{A}B$, $P(B) = P(AB) + P(\bar{A}B)$

$P(\bar{A}B) = P(B) - P(AB)$

$P(A \cup B) = P(A) + P(B) - P(AB)$

If $A \subset B$, then $P(B) = P(A) + P(\bar{A}B) \geq P(A)$

Conditional Probabilities

$P(A|m) = P(Am)/P(m)$

$P(m|A) = P(Am)/P(A)$

$P(Am) = P(A|m)P(m) = P(m|A)P(A)$

[Ex] A box has 3 white balls and 2 red balls. Remove 2 balls without replacement. Prob. of ball 1 is white and ball 2 is red

P(ball 1 is white) = 3/5;

P(ball 2 is red|ball 1 is white) = 2/4 = 1/2;

P(ball 2 is red) = P(ball 1 is white)xP(ball 2 is red)

= 3/5x1/2 = 3/10

Total Probability and Bayes' Theorem

$C = \{A_1, A_2, \ldots A_N\}$

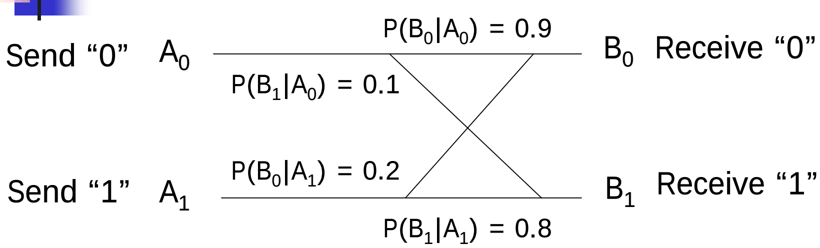
Assume A_i and A_j are mutually exclusive, for any $i \neq j$.

Let B = an arbitrary event

$$P(B) = \sum_{i=1}^{N} P(B|A_i)P(A_i) \equiv \text{total probability}$$

$$P(A_i|B) = P(B|A_i)P(A_i)/P(B)$$

Binary Channel



What are $P(A_0|B_0)$, $P(A_1|B_0)$, $P(A_0|B_1)$, and $P(A_1|B_1)$?

Solution

$P(B_0|A_0)P(A_0) = P(A_0|B_0)P(B_0)$

$P(A_0|B_0) = P(B_0|A_0)P(A_0) / P(B_0)$

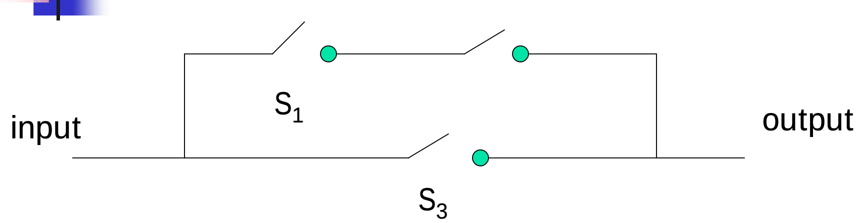
$P(B_0) = P(B_0|A_0)P(A_0) + P(B_0|A_1)P(A_1)$

$= 0.9 \times 0.5 + 0.2 \times 0.5$

$= 1.1 \times 0.5 = 0.55$

$P(A_0|B_0) = 0.9 \times 0.5 / 0.55 = 9/11$

Independent Events



Let event A_i = "switch S_i is closed",
Switches operate independently $P(A_1), P(A_2), P(A_3) = p$

Let event R = "Input is received"
P(R) = ?

Solution

Let A_{12} = "both S_1 and S_2 are closed"

then $\bar{A}_{12}$ = "Either S_1 or S_2 is open"

$P(\bar{A}_{12}) = 1 - P(A_{12}) = 1 - P(A_1)P(A_2)$

R = "both switches S_{12} and S_3 are open"

$$
\begin{aligned}
P(\bar{R}) &= 1 - P(R) = 1 - P(\bar{A}_{12})P(\bar{A}_3) \\
&= 1 - [1 - P(A_1)P(A_2)][1 - P(A_3)] \\
&= 1 - (1 - p^2)(1 - p) \\
&= p^2 + p - p^3
\end{aligned}
$$