MLFMA-Based Quasi-Direct Analysis of Scattering from Electrically Large Targets

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Abstract – The multi-level fast multipole algorithm (MLFMA) is traditionally employed in the context of an iterative matrix solver, in which the MLFMA is utilized to implement the underlying matrix product with $N \log N$ complexity, where $N$ represents the number of unknowns. The total computational complexity of such an approach is order $PN \log N$, where $P$ represents the number of iterations required for iterative-solver (e.g. conjugate gradients) convergence to a desired accuracy. Many electromagnetic-scattering problems are poorly conditioned, and therefore $P$ is often large. In this paper, rather than applying an iterative matrix solver, we perform a matrix product involving the inverse of the impedance matrix. By using the properties of the MLFMA, this process is performed very efficiently for electrically large problems. In particular, numerical experiments indicate that this new formulation (which avoids the iteration count $P$) is often significantly faster than the traditional iterative MLFMA solution, while requiring the same computer memory. The basic theory is described, and several examples are presented.

I. Introduction

The fast multipole method (FMM) and multi-level fast multipole algorithm (MLFMA) [1-6] have received significant attention recently. These new tools markedly accelerate the rate at which a wide class of integral equations can be solved. In particular, in the context of surface and volume integral equations, the MLFMA requires order $N \log N$ complexity and order $N \log N$ memory to perform the matrix products required in an iterative solution of the underlying matrix equation. However, the overall complexity
is order $PN \log N$, where $P$ represents the number of iterations required for the iterative solver (e.g. conjugate gradients [7]) to converge to a desired accuracy.

Many electromagnetic-scattering problems are well known to yield poorly conditioned matrix equations, and therefore the number of iterations $P$ can often be large, especially when high accuracy is required. To address this issue, there has been significant interest in the development of matrix preconditioners [8-14]. These preconditioners are typically implemented in matrix form, and often yield computational enhancement. More recently researchers have sought to understand the underlying properties of the continuous integrodifferential equations. In particular, it has been recognized that the electric-field integral equation (EFIE) is inherently poorly conditioned, with this exacerbated as the spatial sampling rate increases in the discretized problem [11-14]. This has led to continuous-domain preconditioning of the underlying EFIE formulation, prior to discretization, with this yielding encouraging results in the context of a modified MLFMA construct [12-14]. However, this latter formulation is significantly more complicated than a traditional MLFMA formulation [1-6], and therefore it is most appropriate for very high spatial sampling (when the conventional formulation is particularly poorly conditioned).

The aforementioned MLFMA formulations are designed to quickly implement the matrix product $Z_i$, in the context of solving $Z_i v$ (this discussed more fully in Sec. IIA). In particular, the MLFMA effects $Z_i$ with order $N \log N$ complexity, $Z_i$ performed $P$ times in the context of $P$ iterations of the iterative solver (as $i$ is updated). To avoid the problem of encountering large $P$, for a desired accuracy, we here address the problem of directly computing $Z^{-1} v$. As we demonstrate below, this matrix product is also solved using iterative solvers, however the matrix on which such are implemented is far better conditioned than $Z$. This yields substantial improvements in computational complexity, as outlined in the subsequent sections. Moreover, as we elucidate below, the solution of $Z^{-1} v$ for electrically large problems can exploit much of the machinery developed previously for the original MLFMA.
II. Quasi-Direct MLFMA Formulation

A. MLFMA-based matrix partitioning

The integral-equation-based solution of electromagnetic-scattering problems typically yield a discretization of the underlying continuous integrodifferential equations [15]. In particular, the unknown quantity on the surface or within the volume of the target is expanded in terms of a known basis with unknown basis-function coefficients. By imposing boundary conditions on the target, we yield a matrix equation $Zi = v$, where the $N$-dimensional vector $i$ represents the $N$ unknown basis-function coefficients, $v$ is an $N$-dimensional vector representing the incident fields on or in the target, and $Z$ is an $N \times N$ “impedance matrix”, representing the wave interaction between a given set of basis and testing functions (the testing functions enforce boundary conditions) [15].

For electrically large problems, $N$ becomes large and the matrix equation is difficult to solve directly. In particular, a direct inversion of the matrix $Z$ is of order $N^3$ complexity [7]. To circumvent this problem, the matrix equation $Zi = v$ is typically solved iteratively, via such techniques as the conjugate-gradient (CG) method [7]. In an iterative solution we must compute the matrix product $Zi_k$ multiple times, until the unknown vector $i_k$ converges to a desired level of accuracy, as the iteration count $k$ increases. A direct computation of $Zi_k$ is of order $N^2$ complexity, requiring order $N^2$ memory. Therefore, to circumvent this problem researchers have developed the fast-multipole method (FMM) and the multi-level fast multipole algorithm (MLFMA) [1-6].

In the FMM, rather than computing the $N^2$ interactions between the $N$ basis and testing functions directly, we divide these into a set of clusters. The number of clusters $N_c$ is typically much smaller than the number of unknowns $N$. For clusters sufficiently distant [1-6], elements of the matrix product $Zi_k$ can be computed efficiently in terms of cluster-cluster interactions. When the clusters are not sufficiently distant, the associated (basis function)-(testing function) interactions are computed directly. These latter
computations are said to involve “near” terms, while those implemented at the cluster level are termed “far” terms [1-6]. To reduce the number of “near” terms, the multi-level fast multipole algorithm (MLFMA) has been developed [1-6], in which the cluster sizes diminish sequentially. In particular, we first consider large cluster sizes, and all interactions between sufficiently distant clusters are effected via the FMM. Those clusters that are too close at this first level are subsequently divided into eight smaller clusters. For this smaller cluster size, all interactions not accounted for in the previous step are now computed by cluster-cluster computations, again if the inter-cluster spacing is sufficient. This process is repeated until the cluster size is reduced to approximately half a wavelength (such small clusters composed of relatively few basis functions), and all interactions remaining at this level are implemented directly. It has been demonstrated that, using this MLFMA procedure, the matrix product $Z_i$ can be computed with order $N \log N$ complexity.

Based on the above discussion, we can decompose the matrix product $Z_i$ as follows

$$Z_i = (Z_{\text{far}} + Z_{\text{near}}) \hat{i} \quad (1)$$

where $Z_{\text{far}} \hat{i}$ represents the matrix components evaluated cluster-by-cluster, and $Z_{\text{near}} \hat{i}$ represents the remaining portion of the matrix product, implemented directly.

B. Quasi-direct inversion

Rather then utilizing (1) in the context of an iterative solver, our goal is to implement the matrix product $Z^{-1} v$ in terms of $Z_{\text{far}}$ and $Z_{\text{near}}$. This is as opposed to implementing $Z_i$ multiple times within the context of an iterative solver. Using (1), we have

$$Z = Z_{\text{near}} \left( I + Z_{\text{near}}^{-1} Z_{\text{far}} \right) \quad (2)$$

where $I$ is a diagonal matrix with unit elements. We can therefore express the matrix inverse as

$$Z^{-1} = (I - D)^{-1} Z_{\text{near}}^{-1} \quad (3)$$

where $D = -Z_{\text{near}}^{-1} Z_{\text{far}}$. The $p$ norm of the matrix $A$ is defined as [7]
\[ \|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \]  

(4)

where \(x\) is a vector and \(\|x\|_p\) is a traditional vector \(p\) norm [7]. For two matrices \(A\) and \(B\) we also have the property \(\|AB\| \leq \|A\| \|B\|\), where we leave off the subscript \(p\), for simplicity. In addition, it can be demonstrated that [7]

\[ \|A^{-1}\| \leq \frac{1}{1-\|I-A\|} \]  

(5)

from which

\[ \|D\| = \left\| \begin{bmatrix} Z_{\text{near}}^{-1} & Z_{\text{far}} \end{bmatrix} \right\| \leq \frac{\|Z_{\text{far}}\|}{1-\|I-Z_{\text{near}}\|} \]  

(6)

We also expect that \(\|Z_{\text{near}}\| >> \|Z_{\text{far}}\|\), since \(Z_{\text{near}}\) involves generally stronger (basis function)-(testing function) interactions \(\text{vis-à-vis} \ Z_{\text{far}}\) (based on the inter-function spatial distance). Finally, although it is not proven explicitly, based on the above observations we expect that \(\|D\| < 1\), with this observed in all numerical experiments. Using \(\|D\| < 1\), we have the following Neumann series expansion for \((I-D)^{-1}\)

\[ (I-D)^{-1} = \sum_{n=0}^{\infty} D^n \]  

(7)

From (3) and (7), we have

\[ Z^{-1}v = \left[ \sum_{n=0}^{\infty} (-Z_{\text{near}}^{-1} Z_{\text{far}})^n \right] Z_{\text{near}}^{-1} v \]  

(8)

In practice it is observed that the contributions from \((-Z_{\text{near}}^{-1} Z_{\text{far}})^n\) diminish quickly with increasing \(n\), and therefore numerically we need only take a relatively few terms \(M\) from the summation in (8):

\[ Z^{-1}v \approx \left[ \sum_{n=0}^{M-1} (-Z_{\text{near}}^{-1} Z_{\text{far}})^n \right] Z_{\text{near}}^{-1} v \]  

(9)

**C. Physical interpretation**

We now present a simple example that explains in physical terms the meaning of the expansion in (8). In particular, consider the two targets in Fig. 1. The desired basis-
function coefficients associated with the current on targets one and two are represented by $i_1$ and $i_2$, respectively. The associated matrix equation can be expressed as

$$Zi = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = v$$ (10)

where $Z_{11}$ and $Z_{22}$ are sub-matrices accounting for interactions within targets one and two, respectively, while $Z_{12}$ and $Z_{21}$ account for interactions between the targets. For illustrative purposes, we assume that the targets each have dimensions less than one-half wavelength, and that they are separated by multiple wavelengths. Therefore, within the context of the FMM, we assume that each target is contained within a single cluster (see Fig. 1). Under this simplification, we have

$$Z_{\text{near}} = \begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} \end{bmatrix} \quad Z_{\text{far}} = \begin{bmatrix} 0 & Z_{12} \\ Z_{21} & 0 \end{bmatrix}$$ (11)

where, as in (1), $Z = Z_{\text{near}} + Z_{\text{far}}$. This now suggests a simple solution methodology for the matrix equation $(Z_{\text{near}} + Z_{\text{far}})i = v$. A zeroth-order solution for the current ignores the “far” interactions between targets, and simply utilizes the “near” intra-target terms:

$$i^{(0)} = Z_{\text{near}}^{-1} v$$ (12)

We now use the fields generated by $i^{(0)}$ to update the incident fields on the targets, such that the first-order solution satisfies

$$Z_{\text{near}} i^{(1)} = v - Z_{\text{far}} i^{(0)} = v - Z_{\text{far}} Z_{\text{near}}^{-1} v \Rightarrow i^{(1)} = Z_{\text{near}}^{-1} v - Z_{\text{near}}^{-1} Z_{\text{far}} Z_{\text{near}}^{-1} v$$ (13)

If we repeat this process $M$ times, we obtain the approximation

$$i^{(M)} = \left[ \sum_{n=0}^{M-1} (-Z_{\text{near}}^{-1} Z_{\text{far}})^n \right] Z_{\text{near}}^{-1} v$$ (14)

with this expression identical to (9). Therefore, within the context of the simple example in Fig. 1, (14) represents an iterative solution for the induced currents on the two targets. In particular, we first compute the induced currents on the targets as if they are in isolation, neglecting “far” interactions. These currents are used to compute the associated scattered fields, which update the incident fields on the target surfaces (e.g., the updated incident fields on target one are the original incident fields plus the scattered fields from target two). These modified incident fields are used to find an updated version of the induced currents, again analyzing the targets in isolation (using $Z_{\text{near}}$). These revised
induced currents are used to update the target-dependent incident fields (using $Z_{\text{far}}$), and the process is repeated. The physical nature of this process suggests the algorithm will converge, although the convergence rate (required $M$) is dictated by the cumulative resonant character of the two targets.

The example in Fig. 1 is easy to understand because it involves mutual interaction between two targets. This can now be extended to the analysis of a single target. In particular, within the context of the MLFMA, we first only compute the near interactions on the target, characterized by $Z_{\text{near}}$. The fields induced by these approximate currents are used to update the incident fields, through the “far” interactions, characterized by $Z_{\text{far}}$. Once the incident fields are updated, we again revise the currents using the inverse of $Z_{\text{near}}$, and the process is iterated, realizing (9). For a single target, the required iterations $M$ is dictated by the associated target resonant character (here self resonances), which for most realistic targets damp out quickly (particularly perfectly conducting targets). For the perfectly conducting targets considered in Sec. III we find that $M$ is typically quite small.

The physical argument discussed above suggests that (9) will be convergent, thereby suggesting a self consistency with the expansion in (7). Recall that (7) was based on the requirement that $\|D\| < 1$, with this difficult to prove in general mathematically.

D. Numerical implementation

We have implemented the expression in (9) numerically, as follows. The matrix $Z_{\text{near}}$ is sparse, but it is not generally diagonal. Therefore, a direct computation of $Z_{\text{near}}^{-1}$ via an LU-decomposition [7] is computationally expensive. Consequently we define $b_0 = Z_{\text{near}}^{-1}v$, and $b_0$ is computed iteratively via the conjugate-gradient method [7], in terms of $v = Z_{\text{near}}b_0$. Since $Z_{\text{near}}$ is nearly diagonal, with all diagonal elements large [1-6], it is well conditioned and $v = Z_{\text{near}}b_0$ converges quickly (for $b_0$) via a CG iterative solver. Note that the vector $b_0$ is of dimension $N$, with each element corresponding to a particular basis function.
We now define \( b_1 = \mathbf{Z}_{\text{far}} b_0 \), with this computed with \( N \log N \) complexity via the MLFMA; each term in \( \mathbf{Z}_{\text{far}} \) involves “far” interactions, and therefore all interactions are computed on a cluster-by-cluster basis. To complete the computation of \( -\mathbf{Z}_{\text{near}}^{-1} \mathbf{Z}_{\text{far}} \mathbf{Z}_{\text{near}}^{-1} \mathbf{v} \)
we compute \( b_2 = -\mathbf{Z}_{\text{near}}^{-1} b_1 \), again using a CG solver with the well-conditioned \( \mathbf{Z}_{\text{near}} \). This procedure is repeated iteratively to compute all terms \( ( -\mathbf{Z}_{\text{near}}^{-1} \mathbf{Z}_{\text{far}} )^n \mathbf{Z}_{\text{near}}^{-1} \mathbf{v} \) required in (9).

As indicated when presenting results, we have found in practice that we typically need only consider a relatively small \( M \).

Since the number of terms \( M \) required from the sum in (9) is typically much smaller than the number of CG iterations \( P \) required for a traditional iterative MLFMA solution, this procedure often yields significant computational savings. In particular, assume that the average number of “near” neighbors to a given basis function is \( K \), such that the number of non-zero elements in \( \mathbf{Z}_{\text{near}} \) is approximately \( NK \). Then the new formulation requires order \( MP_{\text{near}} KN + (M-1) N \log N \) complexity, where \( P_{\text{near}} < P \) represents the number of CG iterations required to evaluate \( \mathbf{Z}_{\text{near}}^{-1} \). As indicated above, typically \( M \ll P \), and therefore this new quasi-direct MLFMA procedure is generally significantly faster than the traditional iterative approach [1-6]. We term the algorithm “quasi direct” because we directly seek to compute \( \mathbf{Z}^{-1} \mathbf{v} \) (rather than the \( \mathbf{Z}^{-1} \mathbf{Z}^i \) multiple times, as in an iterative MLFMA analysis). However, since \( \mathbf{Z}_{\text{near}}^{-1} \) is computed iteratively, we must re-evaluate \( \mathbf{Z}_{\text{near}}^{-1} \mathbf{v} \) for each new \( \mathbf{v} \), as opposed to a true direct solution, in which we compute \( \mathbf{Z}^{-1} \mathbf{v} \) once (with order \( N^3 \) complexity), and then consider all \( \mathbf{v} \) in \( \mathbf{Z}^{-1} \mathbf{v} \) with minimal additional complexity. We also note that this procedure requires the same computer memory (order \( N \log N \)) as a traditional MLFMA solution [1-6].
III. Example Results

In the sections below we present several results. We first compare the accuracy of the new quasi-direct MLFMA to its traditional iterative counterpart, and then we present experimental results on the computational complexity of the two approaches. In this context it is noted that in previous publications [5,6] we have presented a relatively simple extension of the free-space MLFMA to the case of a target in the vicinity of a half space. Below we therefore present results for targets in free space as well as in the presence of a half space. Moreover, although the procedure in Sec. II is applicable to both perfectly conducting and dielectric targets, we here consider perfect electric conductor (PEC) targets, solved via a combined-field integral-equation formulation [15]. We have used the widely applied RWG basis functions [16], sampled at approximately ten basis functions per wavelength.

A. Model accuracy

We first consider a PEC sphere of one-wavelength radius, situated in free space. The incident angles are defined as $\theta_i = 45^\circ$, $\phi_i = 0^\circ$, and the far-zone radar cross section (RCS) is computed for $0^\circ \leq \theta_s \leq 90^\circ$, $\phi_s = 0^\circ$. In Fig. 2 we present the RCS for VV and HH polarizations, as computed via the traditional MLFMA [1-6] and via the new MLFMA formulation presented here. For this example we required $M=4$ to achieve an accuracy of 0.1%, and the new quasi-direct MLFMA is 40% faster than the traditional iterative formulation. It is seen from Fig. 2 that the two methods are in nearly exact agreement. The iterative MLFMA required $P=32$ iterations to converge to an error of 0.1%.

In the next example we consider a PEC cube, with each side of length two wavelengths. The Cartesian coordinate system is defined by the cube axes, and we consider incidence at $\theta_i = 45^\circ$, $\phi_i = 0^\circ$, and the far-zone radar cross section (RCS) is computed for $0^\circ \leq \theta_s \leq 90^\circ$, $\phi_s = 0^\circ$. For this example $M=4$ to achieve an accuracy of
0.1%, and the new quasi-direct MLFMA is again 40% faster than the traditional iterative formulation. From Fig. 3 we note that the two methods are in nearly exact agreement. The iterative MLFMA required $P=33$ iterations to converge to an error of 0.1%.

In the next example we consider a PEC sphere of one-wavelength radius situated in vacuum, with its center 1.2 wavelength above a lossy half space characterized by complex permittivity $\varepsilon_r = 5 - j0.02$ and conductivity $\sigma = 0.01$ S/m. The incident angles are defined as $\theta_i = 45^\circ$, $\phi_i = 0^\circ$, and the far-zone radar cross section (RCS) is computed for $0^\circ \leq \theta_s \leq 90^\circ$, $\phi_s = 0^\circ$. From Fig. 4 we again note that the iterative and quasi-direct MLFMA are in near exact agreement, with the latter requiring 75% less computational time than the former. In this example $M=7$, and the iterative MLFMA required $P=52$ iterations to converge to an error of 0.1%.

B. Computation time

The previous three examples are representative of numerous numerical experiments we have considered, in which excellent agreement has been demonstrated between the iterative and quasi-direct MLFMA solutions. We have also discussed relative computational complexity, with this issue now addressed further. In particular, we consider scattering from a PEC sphere in free space. We fix the spatial sample rate at approximately eight basis functions per wavelength, and increase the number of unknowns $N$ by increasing the frequency. We have chosen this sampling rate because it represents a regime in which traditional matrix preconditioners add minimal value [10]. In Fig. 5 we plot the computation (CPU) time as a function of the number of unknowns $N$, for the traditional iterative MLFMA and for the new quasi-direct MLFMA. The results were computed on a Pentium III personal computer operating at a 1 GHz clock speed, with 4 Mbytes of memory. The results in Fig. 5 demonstrate that as $N$ grows the computational savings accrued by the quasi-direct approach is more substantial. For example, for $N=53,508$, the quasi-direct MLFMA requires approximately 38% the CPU time as the traditional MLFMA, to achieve the same level of accuracy (0.1% error). For
N=53,508 the iterative MLFMA required $P=26$ iterations, with the quasi-direct MLFMA required $M=6$ terms in (9), with this difference responsible for the CPU disparity. The memory (RAM) required for these computations is identical. We have seen a similar level of acceleration in all targets considered to date (in free space as well as in the vicinity of a half space).

Figure 5 underscores an important characteristic of the new formulation. In particular, we depict the number of MLFMA levels used as a function of $N$, where here there are regions of three, four and five levels. For a fixed number of levels, as $N$ increases, the minimum MLFMA cluster size (in wavelengths) also increases. When the algorithm transitions from $m$ to $m+1$ MLFMA levels the minimum cluster size typically diminishes discontinuously. The new algorithm efficiency is dictated by the percentage of (basis function)-(testing function) interactions treated as “near”. This number increases as the smallest MLFMA cluster size increases. Therefore, with regard to Fig. 5, we note that the CPU time increase with increasing $N$, for a fixed number of MLFMA levels (because the minimum cluster size is increasing with in $N$, in this regime). When there is a transition from $m$ to $m+1$ levels, the CPU time drops, since the number of “near” interactions diminishes (due to a reduction in the minimum MLFMA cluster size). Therefore, there is still opportunity for further algorithmic optimization, in the definition of the “near” interactions, with increasing $N$.

IV. Conclusions

We have presented a new method for solving the matrix equations associated with electrically large scattering problems. Similar matrix equations arise in radiation problems. Such problems have traditionally been solved via iterative matrix solvers, in which one repeatedly computes a matrix product $Zi$ between the fixed impedance matrix $Z$ and the iteratively updated unknown basis-function coefficients $i$. This matrix product can be implement with $N \log N$ complexity via the MLFMA [1-6], ultimately yielding an algorithm of $PN \log N$ complexity, for $P$ iterations of the iterative solver. In practice $P$ is
often large, and therefore we have developed a technique to solve the matrix product $Z^{-1}v$ directly (where $v$ is the known driving vector).

We have demonstrated that the matrix product $Z^{-1}v$ can be implemented efficiently using the tools associated with the traditional MLFMA [1-6], while avoiding iterative solutions involving the often poorly conditioned matrix $Z$. As implemented, the matrix product $Z^{-1}v$ does involve an iterative matrix solver, but now involving a well conditioned, sparse and nearly diagonal matrix, thereby yielding computational efficiency. Since we seek the direct matrix product $Z^{-1}v$, with this partially involving an iterative solver, we term the algorithm “quasi direct”.

We have implemented the new formulation for perfectly conducting targets in free space and in the vicinity of a half space. The results of the new model are in virtually exact agreement with those computed via the traditional MLFMA. However, the new formulation is often significantly more efficient computationally, especially as the number of unknowns $N$ increases. We also note that the basic construct presented here can be extended to more-complex targets, composed of multiple parts [17].

With regard to future research, the basic formulation developed here exploits the underlying properties of the MLFMA, and therefore it is also applicable to surface and volume integral equations for scattering from dielectric targets. Therefore the next obvious direction is to extend the formulation to the case of such targets. In addition, for all examples we have considered to date the new formulation has always been more efficient computationally than the original MLFMA. However, this issue requires a more-exhaustive study, to assure the relative merit of this new formulation, across a wide range of target types. In particular, the physical argument presented in Sec. IIC suggests that the required $M$ in (9) will be dictated by the resonant character of the target under study. Therefore, it is of significant interest to consider dielectric targets, which are often far more resonant than the PEC targets considered here.
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Figure Captions

Figure 1. Iterative solution for scattering from two targets. Each target is enclosed in a single FMM cluster. The zeroth-order solution (top) ignores multiple interactions between the targets (i.e. it is based only on $Z_{\text{near}}$). The first-order solution (middle) uses the zeroth-order induced currents to compute scattered fields (via $Z_{\text{far}}$), these added to the incident fields. Using these updated incident fields, the targets are again analyzed in isolation. The second-order solution (bottom) uses the first-order induced currents to compute the scattered fields on the targets, with these used to update the incident fields. The second-order currents are solved for by again treating the targets in isolation, with the revised incident fields. This process repeats until the induced currents converge.

Figure 2. Radar cross section (RCS) for VV and HH polarizations, as computed via the traditional MLFMA [1-6] and via the new MLFMA formulation presented here. The target is a PEC sphere of one-wavelength radius situated in free space, the incident angles are defined as $\theta_i = 45^\circ$, $\phi_i = 0^\circ$, and the far-zone RCS is computed for $0^\circ \leq \theta_s \leq 90^\circ$, $\phi_s = 0^\circ$.

Figure 3. Radar cross section (RCS) for a perfectly conducting cube in free space, with each side of length two wavelengths. The Cartesian coordinate system is defined by the cube axes, and we consider incidence at $\theta_i = 45^\circ$, $\phi_i = 0^\circ$, and the far-zone RCS is computed for $0^\circ \leq \theta_s \leq 90^\circ$, $\phi_s = 0^\circ$. Results are computed using the new quasi-direct MLFMA formulation presented here, and using the traditional iterative MLFMA [1-6].

Figure 4. Radar cross section (RCS) of a perfectly conducting sphere of one-wavelength radius situated in vacuum, with its center 1.2 wavelengths above a lossy half space characterized by complex permittivity $\varepsilon_r = 5 - 0.02 j$ and conductivity $\sigma = 0.01 \text{ S/m}$. The incident angles are defined as $\theta_i = 45^\circ$, $\phi_i = 0^\circ$, and the far-zone radar cross section (RCS) is computed for $0^\circ \leq \theta_s \leq 90^\circ$, $\phi_s = 0^\circ$. Results are computed using the new
quasi-direct MLFMA formulation presented here, and using the traditional iterative MLFMA [1-6].

**Figure 5.** Computation time (seconds) required to compute the scattered fields from a perfectly conducting sphere in free space, as a function of the number of unknowns $N$. We fix the spatial sample rate at approximately ten basis functions per wavelength, and increase the number of unknowns $N$ by increasing the frequency. Results are shown for the traditional iterative MLFMA [1-6] and for the new quasi-direct MLFMA presented here. We denote the number of MLFMA levels used, as a function of $N$. 
Iteration $m=0$

The targets are analyzed in isolation.

Iteration $m=1$

There is a 1st-order "far" interaction between the targets.

Iteration $m=2$

There is a 2nd-order "far" interaction between the targets.

Figure 1
Figure 2
Figure 3
Figure 4

Scattering angle $\theta$ [deg]

RCSvv [dBsm], RCSvh [dBsm]

$Z^{-1}v$ MLFMA (VV)

$Z^{-1}v$ MLFMA (HH)

Zi MLFMA (VV)

Zi MLFMA (HH)

Figure 4
Figure 5

The graph compares CPU time (in seconds) against the number of unknowns for different levels of the MLFMA method. It shows two curves, one for $Z_i$ MLFMA and another for $Z^{-1}v$ MLFMA. The x-axis represents the number of unknowns, and the y-axis represents CPU time. The graph highlights the performance at 3 levels, 4 levels, and 5 levels of the MLFMA method.