New MLFMA Formulation for Closed PEC Targets in the Vicinity of a Half Space

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Abstract – The multi-level fast multipole algorithm (MLFMA) is applied to the problem of scattering from a closed perfect electric conductor (PEC) in the presence of a half space. The combined-field integral equation (CFIE) employs a new electric-field integral equation (EFIE) formulation, robust to a high basis-function sample rate, relative to wavelength. The new EFIE formulation is discussed, as is its implementation in the context of a CFIE MLFMA analysis for closed PEC targets. Several example results are presented, in which comparisons are made to traditional MLFMA formulations.

I. Introduction

Integral-equation formulations have been used for several decades in the analysis of scattering from targets. In this context researchers have developed electric-field integral equations, magnetic-field integral equations, and combined-field integral equations [1] (EFIE, MFIE and CFIE, respectively). These integral equations are typically solved by expanding the unknown surface currents in a basis, where the triangular-patch basis functions introduced in [2] are a widely utilized basis-function choice. The $N$ basis-function coefficients are solved for by imposing boundary conditions (implemented with an appropriate set of testing functions), yielding an $N \times N$ matrix equation, traditionally solved in the context of the method of moments (MoM) [3]. A direct MoM solution requires order $N^2$ complexity to fill the $N \times N$ matrix, order $N^2$
memory for matrix storage, and order $P \cdot N^2$ complexity to solve the matrix equation, where $P$ reflects the number of iterations required for an iterative solver, such as the conjugate-gradient (CG) method [4]. To mitigate these computational and memory requirements, researchers have recently developed the multi-level fast multipole algorithm (MLFMA) [5-10], this requiring order $N \log N$ memory and order $P \cdot N \log N$ computational complexity.

One of the remaining challenges in the analysis of such integral equations involves the required number of CG iterations, $P$. To address this problem there has been significant work directed toward the development of preconditioners. Most of these preconditioners have been implemented in the discrete (matrix) domain. In particular, let $Zi\!\!=\!\!v$ represent the matrix equation, where $Z$ represents the $N \times N$ “impedance” matrix, $i$ is an $N \times 1$ vector representative of the unknown basis functions, and $v$ is also an $N \times 1$ vector, representing the driving function or incident waves (within the context of the MLFMA [5-10], the full matrix $Z$ is not evaluated explicitly). Most preconditioners have been implemented in the context of a $N \times N$ matrix $M$, which is multiplied to both sides of the matrix equation. In particular, a properly designed matrix $M$ yields improved iterative convergence of the matrix $MZi\!\!\!=\!\!Mv$. Several different techniques have been considered for design of $M$, and these have often resulted in improved solution convergence [11-13].

More recently researchers have sought to understand the limitations in the underlying continuous integral equations that manifest poor conditioning in the subsequent discrete (matrix) equation. In this context it has been recognized that the “hyper-singular” characteristics of the EFIE are responsible for much of the poor conditioning, particularly when the basis-function sample rate is high relative to wavelength [14-17]. If $T$ is used to represent the EFIE operator, it has been recognized that improved matrix conditioning is realized when employing the modified underlying operator $T^2$ (i.e. the EFIE matrix equation is based on $T^2$ rather than $T$) [15-17].

In this paper we discuss techniques by which $T^2$ can be discretized, yielding a matrix equation. The basic ideas in this context are discussed in [17], where we here
focus on the new formulation in the context of a MLFMA formulation [5-10]. As we discuss more fully in Sec. II, discretization of $T^2$ is implemented based on an intermediate space, used to implement the right operator in $T^2=TT$. By choosing this intermediate space appropriately, the right operator is implemented in a form very similar to the original MLFMA [5-10]. The left operator $T$ in $T^2=TT$, representative of transitioning from the intermediate space to the testing-function space, is here evaluated approximately by only accounting for “near” interactions. This is shown to yield both computational efficiency and highly accurate results.

Having modified the EFIE, we must now consider its use in the context of a CFIE formulation employed for closed PEC targets. In particular, recall that the CFIE is designed to eliminate the resonances present in isolated EFIE and MFIE formulations (these present when the EFIE and MFIE are not combined). We demonstrate below that the modified EFIE (M-EFIE) operator $T^2$ is characterized by both EFIE and MFIE resonances, and therefore a direct combination of the M-EFIE and MFIE does not eliminate internal MFIE resonances. We therefore develop a modified form of $T^2$ that eliminates MFIE resonances while still retaining improved numerical conditioning, with this employed in the subsequent CFIE.

The remainder of the text is organized as follows. In Sec. II we summarize the new EFIE formulation, with a focus on its numerical implementation in the context of a MLFMA analysis. Example results are presented in Sec. III for closed PEC targets in the presence of a half space, with comparisons to traditional CFIE and EFIE implementations. The work is summarized in Sec. IV, where suggestions for future research are also discussed.

II. Modified Electric-Field Integral Equation

A. Review of traditional EFIE

Consider a closed perfect electric conductor (PEC) situated in free space. We extend the formulation to the half-space case in Sec. IIE. The integral-equation analysis
of scattering from such a target is typically implemented in terms of a combined-field integral equation (CFIE), this a weight summation of an electric-field and magnetic-field integral equation (EFIE and MFIE, respectively) [1]. As is well known, it is the EFIE component that often manifests numerical complications [14-17], and therefore we initially focus on this term. For the free-space case, the EFIE is expressed as

$$ T \mathbf{J} = M^i $$

(1)

where \( \mathbf{J} \) is the normalized surface current \( \mathbf{J} = \eta \mathbf{n} \times \mathbf{H} \), where \( \eta = \sqrt{\mu / \epsilon} \) is the wave impedance, \( \mu \) and \( \epsilon \) are respectively the permeability and permittivity, \( \mathbf{n} \) is a unit vector normal to the target surface, \( \mathbf{H} \) is the magnetic field, \( M^i = -\mathbf{n} \times \mathbf{E}^i \), and \( \mathbf{E}^i \) represents the incident electric field. The EFIE operator \( T \) is defined as \( T = T_s + T_h \) where [17]

$$ T_s \mathbf{J} = -jk \mathbf{n} \times \int_{S} \mathbf{J}(r')G(r,r')dS' $$

(2)

$$ T_h \mathbf{J} = -\frac{1}{jk} \mathbf{n} \times \int_{S} \nabla' \cdot \mathbf{J}(r')\nabla G(r,r')dS' $$

(3)

\( k = \omega \sqrt{\mu \epsilon} \) and \( \omega \) represents the angular frequency. The free-space Green’s function is represented by

$$ 4\pi G(r,r') = \frac{\exp (-jk|r-r'|)}{|r-r'|} $$

(4)

The form of the integral equation in (1) is known to be poorly conditioned, especially when the basis-function sample rate is fine relative to wavelength [14-17]. This characteristic is linked to the hyper-singular component of the operator, \( T_h \). This poor conditioning results in relatively slow convergence of iterative (conjugate gradient [17]) solvers for a discretized form of (1).

**B. Modified EFIE formulation**

It has been demonstrated that a modified form of (1), characterized by improved conditioning, is manifested as [15-17]

$$ TT \mathbf{J} = TM^i $$

(5)
A detailed justification of (5) can be found in [17]. Here we focus on the computational implementation of (5), particularly in the context of the multi-level fast multiple algorithm (MLFMA) [5-10]. Utilizing the relationship \( T = T_s + T_h \), we have [17]

\[
TTJ = T^2J = (T_s^2 + T_s T_h + T_h T_s)J
\]  

(6)

Assume that we expand the unknown current \( J \) in terms of RWG basis functions \( f_i \) [2]

\[
J(r') = \sum_i a_i f_i(r')
\]  

(7)

At issue is how one utilizes the expansion in (7), in the context of the set of double operators in (6). We must also couple this process with the set of testing functions [3]. This is effected in the following two-step process.

The operator \( T_s \) is a smoothing operator, in the sense that it integrates the basis functions \( f_i \). Note that the range of the operators \( T_s J \) and \( T_h J \) is over the entire surface \( S \) of the target under consideration. In the context of the expansion in (7), the operator \( T_s \) and \( T_h \) operate on the basis functions \( f_i \) used for the current expansion. We now implement the composite operations \( T_s^2 J, T_h T_s J \) and \( T_s T_h J \) as follows. We expand the result of \( T_s J \) in a separate basis, \( b_i \), where \( b_i \) exist on the target surface, like \( f_i \). We call the space spanned by \( b_i \) an intermediate space \( S_s \). We similarly expand the result of \( T_h J \) in a basis \( g_i \), where \( g_i \) spans an intermediate space \( S_h \). Now, to implement \( T_s^2 J \) and \( T_s T_h J \), we respectively operate \( T_s \) on the result of \( T_s J \) and \( T_h J \), after the latter are first projected onto the respective spaces \( S_s \) and \( S_h \). The operations \( T_s^2 J \) and \( T_s T_h J \) are therefore implemented by effecting two operations, on two different spaces. We first implement \( T_s J \) and \( T_h J \) on the space spanned by the RWG basis functions \( f_i \), the results of which are projected onto \( S_s \) and \( S_h \), respectively. This is followed by second operation, using \( T_s \) now operating on the spaces spanned by \( S_s \) and \( S_h \). We note from (2) that the operation \( T_s J \) results in a vector in the direction \( n \times J \), and since \( J \) is expanded in terms of \( f_i \), we expand \( T_s J \) in terms of the basis \( n \times f_i \) (i.e., the space \( S_s \) is spanned by the functions \( b_i = n \times f_i \)). We have also used the bases \( g_i = n \times f_i \) to span the space \( S_h \). Since the expansion and intermediate spaces for \( T_s \) and \( T_h \) are the same, we perform the operation \( T_s + T_h \) in one composite step, followed by the subsequent operation by \( T_s; T_s(T_s + T_h) \). As discussed further below, the operation of \( T_s + T_h \)
on the expansion basis, followed by testing onto the intermediate spaces $S_s$ and $S_h$, are implemented efficiently here via the MLFMA (see Sec. IID).

In the context of $T_h T_s J$, a similar mapping of $T_s J$ to an intermediate space is more complicated, because the second operator $T_h$ effects a differentiation of the intermediate basis, and therefore the required intermediate space would require a more-sophisticated basis. In lieu of this, we perform a simpler two-step process for $T_h T_s J$. In particular, we consider

$$T_h T_s J = P_{hs} Q_{hs} J$$

(8)

where

$$P_{hs} \rho = n \times \int_S \rho(r') \nabla G(r, r') dS'$$

(9)

$$Q_{hs} J = n \cdot \int_S J(r') \times \nabla G(r, r') dS'$$

(10)

Note that $Q_{hs} J$ yields a scalar quantity, so the intermediate space $S_{hs}$ between $Q_{hs}$ and $P_{hs}$ only requires a scalar basis. Again assuming that $J$ is expanded in an RWG basis, the scalar intermediate space $S_{hs}$ is spanned by scalar functions $\phi$ over each triangle of the RWG mesh. The scalar $\phi$ have unit value along the common edge, going linearly to zero at the free nodes.

The above procedure, in which we utilize intermediate spaces, results in a two-step process, in which we first effect $T_s J$ and $T_h J$, with $J$ expanded in $f_i$. The results of $T_s J$ and $T_h J$ are then projected onto the space spanned by $n \times f$. Similarly, we realize $Q_{hs} J$ with $J$ expanded in $f_i$, with $Q_{hs} J$ projected into the space spanned by $\phi_i$. Each of these first steps in the two-step procedure can be implemented via the MLFMA, with minimal modifications to the MLFMA construct. We now consider the second step, in which the results of the first are subjected to an additional operator ($T_s$ and $P_{hs}$, for the respective terms in (6)).

After performing the mapping to the respective intermediate spaces $S_s$, $S_h$ and $S_{hs}$, interaction with the subsequent operator ($T_s$ and $P_{hs}$) and the testing functions can also be
implemented via the MLFMA. However, recall that the MLFMA involves “near” and “far” terms, the “near” terms effected via a nearly diagonal matrix $Z_{\text{near}}$ [5-10]. In the work reported here, the operations from the expansion functions (7) to the respective intermediate spaces are effected via a complete MLFMA formulation. However, for the second step, from intermediate space to testing functions, we only utilize the respective terms $Z_{\text{near}}$. This is done to simplify overall computational complexity, and we demonstrate in Sec. III encouraging results. The MLFMA component is discussed further in Sec. IID.

C. Combined-field integral equation for closed structures

It is well known that the EFIE formulation of scattering from closed PEC targets is beset by artifacts induced by internal resonances. We therefore have considered and implemented a combined-field integral equation (CFIE) [1]. In particular, we consider the integral equation

$$T^2 J + \alpha(\frac{1}{2} + K)J = TM^i + \alpha\eta J^i$$

(11)

where $J^i = n \times H^i$, $H^i$ is the incident magnetic field, and $\alpha$ is a prescribed real constant. The operator $K$ comes from the magnetic-field integral equation (MFIE), and is expressed as

$$KJ = -n \times \int_S \nabla \times J dS'$$

(12)

It is well known that the MFIE is not characterized by the numerical difficulties associated with the EFIE, and therefore we do not perform regularization on the MFIE component of the CFIE.

Assume that the eigenvalues of the discretized operator $K+1/2$ are given by $|\lambda_L| < |\lambda_{L-1}| < \ldots < |\lambda_1|$. Using the identity [18]

$$T^2 = (K + \frac{1}{2})(K - \frac{1}{2}) = (K + \frac{1}{2})[(K + \frac{1}{2}) - 1] = (K + \frac{1}{2})^2 - (K + \frac{1}{2})$$

(13)
the eigenvalues of the discretized $T^2 + \alpha(\frac{1}{2} + K)$ are given by

$|\lambda_L(\lambda_L + \alpha - 1)| < |\lambda_{L-1}(\lambda_{L-1} + \alpha - 1)| < \ldots < |\lambda_1(\lambda_1 + \alpha - 1)|$. Using the ratio of the largest to smallest eigenvalues as a definition of the condition number $CN$ [4], the condition number of the discretized $T^2 + \alpha(\frac{1}{2} + K)$ is given by

$$CN = \frac{|\lambda_L(\lambda_L + \alpha - 1)|}{|\lambda_1(\lambda_1 + \alpha - 1)|}$$  \hspace{2cm} (14)

We note that for $\alpha = 1$ the condition number is the square of that for $(\frac{1}{2} + K)$ alone. In the limit of large $\alpha$ the condition number in (14) converges to that for $(\frac{1}{2} + K)$. In the work reported here we have chosen $\alpha = 3$, this yielding conjugate-gradient convergence in approximately 20% fewer iterations, for a given error criterion.

D. MLFMA implementation

As indicated in Sec. IIB, we have implemented the MLFMA for the cumulative operator $T_s + T_h$. In particular, the MLFMA is employed to account for the operation of $T_s + T_h$ on the basis functions $f_i$, the results of which are mapped to the subspace spanned by $n \times f_i$ (these latter functions act as testing functions in the first operator step). A similar MLFMA procedure is implemented in the context of (10), where here the expansion functions are again $f_i$, with the scalar basis $\phi_i$ constituting the testing functions. The second step can also be implemented via the MLFMA, this characterized by respective operations with $T_s$ and $P_{hs}$, followed by inner products with the testing functions. Rather than performing a full MLFMA analysis for this step, we simply utilize the associated “near” MLFMA terms, characterized by a sparse matrix $Z_{near}$ [5-10]. Note that the $Z_{near}$ matrices required for $T_s$ and $P_{hs}$ are distinct.

There is an important distinction between the MLFMA implementation on (11), and the MLFMA implementation as applied to the traditional CFIE operator.
In the traditional CFIE the MLFMA treatment of the EFIE and MFIE components is handled simultaneously. For the formulation presented here, the MFIE term is characterized by direct interactions between expansion and testing functions, with the intervening MFIE operator \( \frac{1}{2} \mathbf{J} (1 + \mathbf{K}) \). The EFIE term is characterized by interactions between the expansion functions and *intermediate* testing spaces \((S_s, S_h\) and \(S_{hs}\)). Therefore, within the context of the MLFMA, the EFIE and MFIE components are analyzed separately. Moreover, within the context of the EFIE, we have the additional interactions with the appropriate matrices \(Z_{near}\) (see previous paragraph). In practice, we have found that the stabilized CFIE, with the modified EFIE requires approximately twice the CPU per conjugate-gradient (CG) iteration than the traditional MLFMA CFIE implementation [5-10]. Consequently, the new formulation is only beneficial when the number of CG iterations required to achieve a prescribed error criterion is less than one-half that require from the traditional MLFMA CFIE. As we discuss in Sec. III, we have found such acceleration in many examples.

A final comment is required concerning the operator \( \mathbf{T}^2 \). For a closed PEC target it is well known that the operators \( \mathbf{T} \) and \((1/2+\mathbf{K})\) are individually beset by internal resonances (EFIE and MFIE resonances, respectively). The traditional CFIE formulation is designed to mitigate these internal resonances [1]. In particular, the EFIE and MFIE resonances are typically distinct, and therefore by combining the two the composite integral equation (CFIE) is characterized by neither EFIE nor MFIE resonances. However, in the new formulation presented here the EFIE portion of the CFIE is composed of \( \mathbf{T}^2 \) rather than \( \mathbf{T} \). This operator is beset by the resonances associated with \( \mathbf{T} \) (EFIE resonances), and from (13) we note that the \((K+1/2)\) term also yields MFIE internal resonances. Therefore, a CFIE based on a direct combination of \( \mathbf{T}^2 \) and \((K+1/2)\) is undermined by MFIE resonances (as discussed further when presenting results, in Sec. III). We note that \( \mathbf{T} \) is a function of the wavenumber \( k \) in which the PEC target resides, with this dependence denoted explicitly as \( \mathbf{T}(k) \). When implementing the aforementioned numerical solution, we effect the approximation \( \mathbf{T}^2=\mathbf{T}(k)\mathbf{T}(k)\approx\mathbf{T}(k_1)\mathbf{T}(k) \), where \( k_1 \) is
distinct from $k$. In the work reported here $k_1$ is chosen as $k_1 = k - 0.5k$. It is important to note that $T(k_1)T(k)$ does not satisfy the identity in (13), and therefore the MFIE resonances are eliminated from $T(k_1)T(k)$. The CFIE is therefore characterized by no internal resonances (in the numerical experiments undertaken thus far). We note that the choice of $k_1$ employed here has not been optimized.

E. Half-space implementation

The formulation summarized in the previous sections was based on the free-space Green’s function in (4). In the results reported here we consider targets in the presence of a half space, of interest for many remote-sensing applications. This is performed by implementing an asymptotic analysis of the dyadic half-space Green’s function, from which we obtain a direct-radiation term and a term associated with reflection at the half-space interface [9,10]. The former is characterized via the free-space Green’s function, while the latter is characterized via a weighted version of the free-space Green’s function. The latter weighting is representative of the associated half-space reflection coefficient [9,10]. Therefore, within the context of this asymptotic representation, both the direct and reflected components are characterized via the free-space Green’s function, from which one effects a modified form of the traditional MLFMA implementation [9,10]. The asymptotic form of the half-space Green’s function is used for the “far” MLFMA terms, while a rigorous Green’s-function evaluation is performed for the “near” terms (implemented via the complex-image technique). Using this construct we obtain a relatively efficient MLFMA formulation, while achieving a high degree of numerical accuracy.

III. Example Results

The principal utility of the new EFIE formulation is in the context of problems for which the spatial discretization rate for the induced surface currents is fine relative to wavelength [14-17]. It is for such problems that the traditional EFIE and CFIE become less stable, requiring increased iterations for iterative solvers to achieve a prescribed accuracy. We therefore address such issues in the context of the results presented below.
In all results discussed below the iterative solver is implemented via the conjugate-gradient (CG) method [4].

In all the subsequent examples the lower half space is characterized by relative permittivity $\varepsilon_r = 5 - 0.2j$ and conductivity $\sigma = 0.01$ S/m. Moreover, in all examples the targets are situated above the lower half space (in vacuum), although we have demonstrated previously that the MLFMA formulation implemented here is also applicable for buried targets [9,10] (our previous MLFMA results only considered the traditional CFIE formulation).

In the first example we consider a sphere of $0.2\lambda$ radius (the computations are performed for $\lambda = 1.5$ m), and the bottom of the sphere is 10 cm above the half-space interface. We consider plane-wave excitation at the angle $\theta = 0$ from the normal. In Table 1 we tabulate the number of CG iterations required to achieve an error of 0.1%, with three results shown. The CFIE results are representative of the traditional combined-field formulation [1]; the N-CFIE are representative of the new CFIE formulation presented here, in which all interactions associated with the second $T$ operator (in $T^2$) are implemented rigorously; and M-CFIE is a modified and simplified form of the new CFIE formulation, in which the second $T$ operator is implemented using only the near terms $Z_{near}$. Two numbers are given for each method and for each number of unknowns $N$, with these reflective of the number of CG iterations required for the two plane-wave polarizations. We see that the required CG iterations are relatively independent of polarization. We also note that the simplified M-CFIE provides stability comparable to the rigorous N-CFIE, and for large $N$ the required CG iterations are significantly reduced relative to the traditional CFIE.

Concerning the results in Table 1, each iteration in M-CFIE is approximately twice as expensive computationally vis-à-vis the simpler CFIE. However, for the examples in Table 1 we see that the CFIE always requires at least three-times more CG iterations to achieve the same error criterion, and therefore the M-CFIE provides improved overall computational efficiency. For the very large sample rate, $N=6402$, the
M-CFIE is approximately fifty times more efficient than the traditional CFIE. In Table 1, the \( N=642 \) corresponds to approximately 18.75 RWG basis functions \([2]\) per wavelength, while the \( N=6402 \) corresponds to approximately 60 basis functions per wavelength.

We next consider a PEC cube with sides of length \( 0.4\lambda \) (\( \lambda =1.5 \)), situated with two of its sides parallel to the half-space interface, at a height of 10 cm. The cube is characterized by eight corners, and we have found its CG convergence rate to be considerably slower than that of the (smooth) sphere. The results for this case are tabulated in Table 2. While the N-CFIE and M-CFIE results are far more stable with increasing \( N \) than the CFIE, we note that for this example the overall computation time of the M-CFIE is larger than that of the CFIE, up to \( N=3042 \), corresponding to a spatial sample rate of 28.5 basis functions per wavelength. This example underscores that the added complexity of the M-CFIE formulation is most useful for a high spatial sampling (recall that the M-CFIE is approximately twice as expensive computationally then the original CFIE, per CG iteration). In this example the plane-wave incidence angle is \( \theta_i = 0 \) and \( \phi_i = 0 \), where the coordinate system is defined by the axes of the cubic target.

In the next example we consider a sphere of radius \( 0.436674570\lambda \), for which the lowest-order MFIE mode is resonant (in the computations \( \lambda =1.5 m \)). The target is situated 10 cm above the half-space interface. For this case a MFIE formulation fails, as does an EFIE formulation based on \( T^2=T(k)T(k) \) (see discussion in Sec. IID). Bi-static scattering results are shown in Fig. 1 for incident angles \( \theta_i = 0^\circ \) and \( \phi_i = 0^\circ \), for the CFIE, M-CFIE [based on \( T^2=T(k_1)T(k) \)] and EFIE formulations, using \( N=2982 \) basis functions, corresponding to a sampling rate of approximate 18.6 basis functions per wavelength. The traditional EFIE and CFIE formulations require 185 and 40 CG iterations, respectively, while the M-CFIE required 16 iterations. In all cases the iterative solver is terminated after an error of 0.1% is achieved. We see in Fig. 1 that the accuracy of the M-CFIE is excellent.
This example was selected to examine the effects of internal MFIE resonances (Sec. IID). In particular, in the M-CFIE formulation the operator $T^2$ is approximated as $T^2 = T(k)T(k) = T(k_1)T(k)$, from which we achieved the results in Fig. 1. As expected, when we implemented $T^2 = T(k)T(k)$ directly, the CG solution was divergent (due to the MFIE resonance). Note that the traditional CFIE and EFIE formulations [1] are not undermined by internal MFIE resonances, and therefore this is not an issue for the associated results presented in Fig. 1. However, as indicated above, the traditional CFIE and EFIE formulations are more computationally expensive than the M-CFIE, for this example.

In our final example we consider a sphere of $1\lambda$ radius, situated 10 cm above the half-space interface. The excitation is as in Fig. 1, and here we consider $N=12,264$ (corresponding to a sample rate of approximately 16.5 basis functions per wavelength). To achieve an error of 1%, the M-CFIE required 10 CG iterations, while the traditional CFIE formulation required 32 iterations (the overall computation time of the M-CFIE is approximately two-thirds that of the traditional CFIE). A comparison of the bistatic RCS predicted by the CFIE and M-CFIE are shown in Fig. 2, for which excellent agreement is observed.

IV. Conclusions

A new formulation has been presented for the CFIE analysis of scattering from a closed perfectly conducting target. The focus has been on the EFIE portion of the CFIE, the EFIE having well-known poor conditioning when the basis-function sampling rate increases. Rather then using the traditional EFIE operator $T$, we utilize $T^2$, with appropriate augmentation to account for internal MFIE resonances. The new CFIE formulation has been implemented in the context of the multi-level fast-multipole algorithm (MLFMA), wherein we make an approximation to achieve computational efficiency. In particular, the right-most operator in $T^2 = TT$ is implemented as in a traditional MLFMA analysis [5-10], constituting a map from the basis space to an appropriately defined intermediate space. The second (left) $T$ operator is implemented by
going from the intermediate space to the testing-function space; in this step we only utilize the near interactions embodied by the sparse matrix $Z_{\text{near}}$.

Within the context of the MLFMA, as implemented here, the modified EFIE operator and the MFIE operator are analyzed separately, and then combined. Therefore, each MLFMA CG iteration is approximately twice as expensive computationally as the MLFMA analysis of the traditional CFIE operator (in which the EFIE and MFIE components are handled in one step [5-10]). Therefore, in practice the new CFIE formulation is only beneficial for those cases for which the modified CFIE requires fewer than half as many CG iterations as the traditional CFIE. As demonstrated here, this is the case for examples characterized by a high basis-function sample rate relative to wavelength (see Tables 1 and 2).

The formulation presented here is applicable to targets in the presence of a half space. Within the context of the “far” MLFMA terms, we employ an asymptotic analysis of the half-space Green’s function, while the Green’s function is evaluated rigorously for the “near” terms [9,10]. It has been demonstrated that the asymptotic Green’s function is nearly exact for the source-observation distances characteristic of the “far” terms [9,10].

All of the results considered here are for closed PEC targets. Additional research is required for the case of dielectric targets. For a dielectric target one must achieve a spatial sampling rate that is fine relative to the wavelength in the highest dielectric region (i.e., for a dielectric target of permittivity $\epsilon_r > 1$ residing in vacuum, the basis functions must be sampled finely relative to the wavelength in the $\epsilon_r$ region). In the low-dielectric-constant regions, this implies a high spatial sample rate relative to wavelength. Recall the poor conditioning of the EFIE portion of the integral equation when fine spatial sampling is required. Therefore, it is anticipated that the new EFIE construction may be particularly important in the context of dielectric targets, for which high spatial sampling is always required in low-dielectric-constant regions.
References


Table Captions

**Table 1.** Conjugate-gradient (CG) iterations required to achieve an error of 0.1%, for plane-wave scattering from a PEC sphere of 0.2λ radius (the computations are performed for λ = 1.5 m), with the bottom of the sphere 10 cm above a half-space interface. The lower half space is characterized by relative permittivity ε_r = 5 − 0.2 j and conductivity σ = 0.01 S/m. The CFIE results are representative of the traditional combined-field formulation [1]; the N-CFIE are representative of the new CFIE formulation, in which all interactions associated with the second T operator (in T^2) are implemented rigorously; and M-CFIE is a modified form of the new CFIE formulation, in which the second T operator is implemented using only the near terms Z_{near}. The two numbers given for each method and for each number of unknowns N are reflective of the number of CG iterations required for the two plane-wave polarizations.

**Table 2.** As in Table 1, for a PEC cube with sides of length 0.4λ (λ = 1.5), situated with two of its sides parallel to the half-space interface, at a height of 10 cm.

Figure Captions

**Figure 1.** Bi-static radar cross section for a sphere of radius 0.43667457λ (λ = 1.5 m). The target is situated 10 cm above the half-space interface considered in Table 1. The plane-wave incident angles are θ_i = 0° and φ_i = 0°, and the scattered fields are for φ_s = 0° and 0° ≤ θ_s ≤ 90°. Results are shown for traditional EFIE and CFIE formulations, as well as for the M-CFIE developed in this paper. All results are computed via the MLFMA.

**Figure 2.** Bi-static radar cross section as in Fig. 1, for a PEC sphere of 1λ radius (λ = 1.5 m), situated 10 cm above the half-space interface.
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Table 2
Figure 1
Figure 2