Enhanced Compressed Sensing based on Iterative Support Detection

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Joint work with Yilun Wang

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Notation

- x: sparse signal, has $\leq k$ nonzero entries
- $b = Ax$: CS measurements

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- $b = Ax$: CS measurements

- ℓ_0-problem: $\min \|x\|_0$, s.t. $Ax = b$. Exact recovery needs $m \geq 2k$ for Gaussian A
- ℓ_1-problem: $\min \|x\|_1$, s.t. $Ax = b$. Needs a much bigger m Also called *Basis Pursuit*

Outline

1. Overview
   - The Approach
   - Simple Examples
2. Theoretical Results
   - Summary
   - The Null Space Property
   - Recoverability Improvement
3. Numerical Results
   - Noiseless measurements
   - Noisy measurements
   - A failed case
4. Conclusions

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  T^C: discoveries = correct $\cup$ wrong, out of ℓ_1-norm.
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  $t = |T|$
- Solve

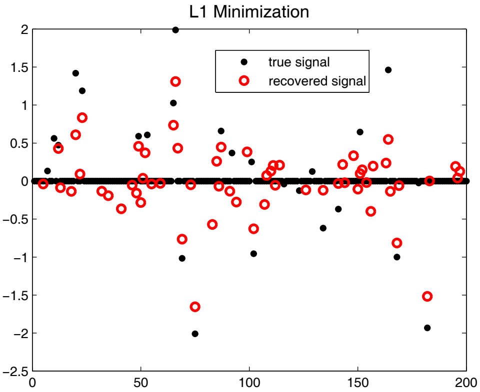
Truncate ℓ_1-problem: $\min_x \|x_T\|_1$, s.t. $Ax = b$.

A Simple Example

Setup:

- $n = 200$, $k = 25$, $m = 2k = 50$, A is Gaussian random

Basis pursuit result: $x^{(1)}$



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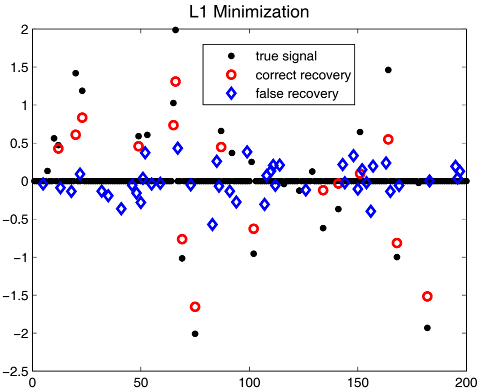

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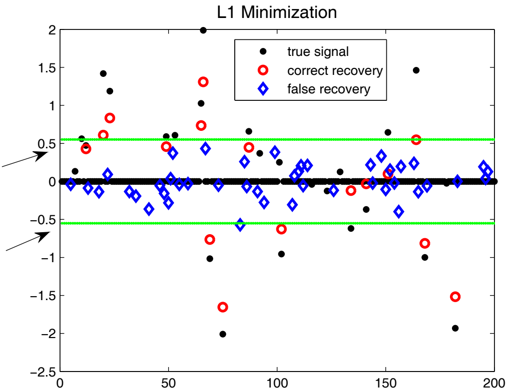
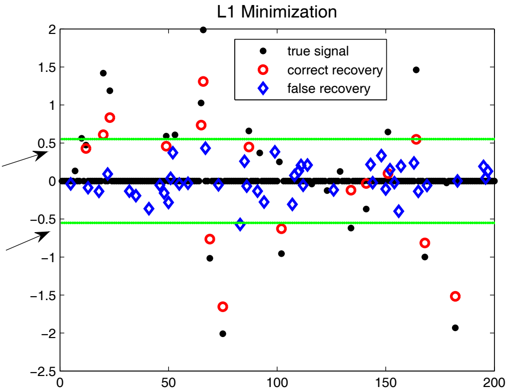


A Thresholding Framework

- Initialize: $j \leftarrow 1$ and $T = \{1, 2, \ldots, n\}$.
- While *not converged* do
  1. Truncated ℓ_1-minimization:
  $$x^{(j)} \leftarrow \min \|x_T\|_1 \quad \text{s.t. } Ax = b.$$
  2. Support detection *by thresholding*:
  $$\epsilon \leftarrow \|x^{(j)}\|_\infty / 3^j,$$
  $$T \leftarrow \{i : |x_i^{(j)}| < \epsilon\}.$$

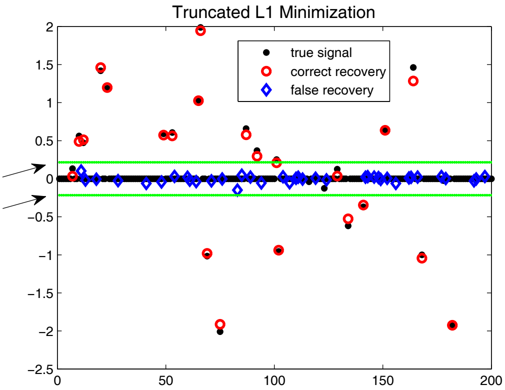
Results of Iterative Thresholding

Basis pursuit result: $x^{(1)}$, threshold $\epsilon = \|x^{(1)}\|_\infty/3$



Results of Iterative Thresholding

Truncated ℓ_1-result: $x^{(2)}$, reduced threshold $\epsilon = \|x^{(2)}\|_\infty / 3^2$



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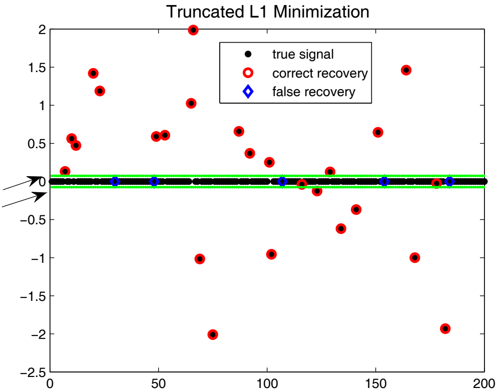
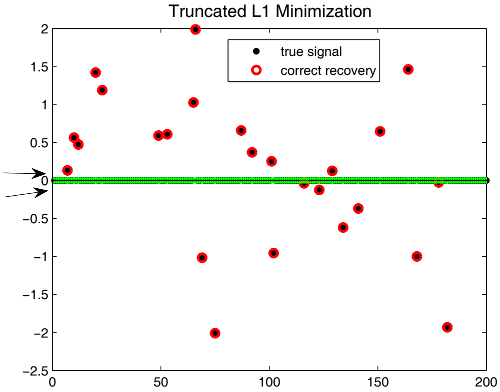
Truncated ℓ_1-result: $x^{(3)}$, reduced threshold $\epsilon = \|x^{(3)}\|_\infty / 3^3$



Results of Iterative Thresholding

Truncated ℓ_1-result: $x^{(4)}$, exact recovery!

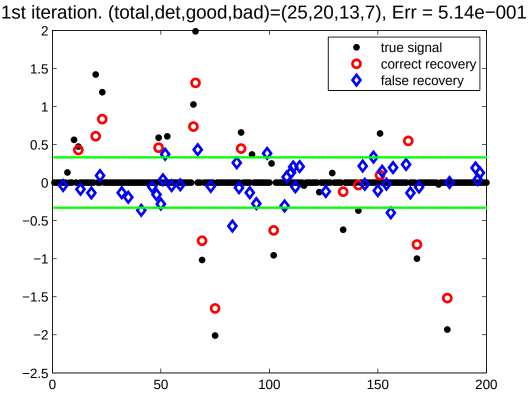


Robustness Test

Try tighter thresholds:

$$\epsilon = \|x^{(j)}\|_\infty / 5^j.$$

$j = 1$, basis pursuit result:



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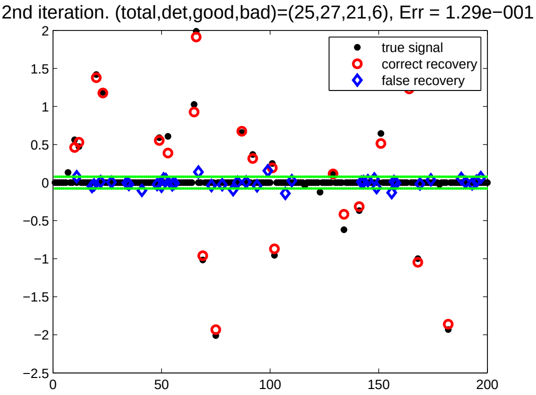

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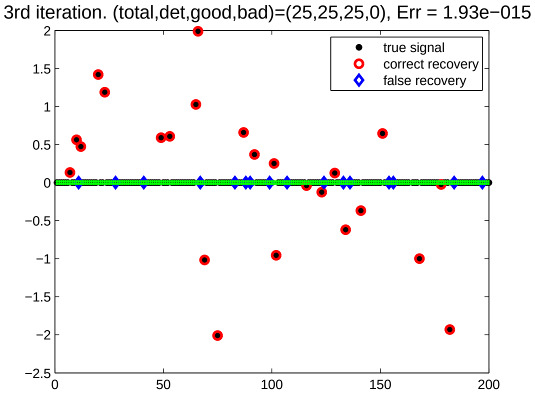
$j = 3$, truncated ℓ_1-minimization result:



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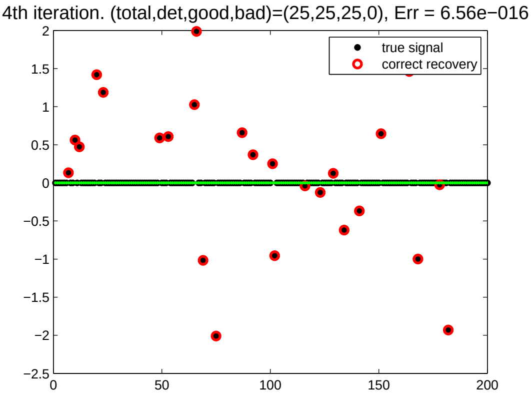

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    compute the size of tail.
  - when to stop?
    tail is zero or small enough.

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We need $\|v_S\|_1 < \|v_{S^C}\|_1$.

- A necessary condition for uniform exact recovery for all $|S|$-sparse signals.

Null Space Property

Definition (Cohen-Dahmen-DeVore and others)

$A \in \mathbb{R}^{m \times n}$ has the *Null Space Property* (*NSP*) with order L and $\gamma > 0$ if

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- NSP is more essential than RIP for basis pursuit (left multiplying A by a nonsingular matrix changes RIP but not NSP)

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Definition (Y.-Wang)

$A \in \mathbb{R}^{m \times n}$ has the *Truncated Null Space Property* (T-NSP) with t, L, and γ, written as T-NSP(t, L, γ), if

$$\|v_S\|_1 \leq \gamma \|v_{T \setminus S}\|_1, \quad \forall S \subset T, |S| \leq L, |T| = t, v \in \mathcal{N}(A).$$

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Theorem (Y.-Wang)

For T given , if A satisfies T-NSP$(|T|, L, \gamma)$ where $\gamma < 1$, then truncated ℓ_1-minimization over the support of T yields an exact recovery.

Recoverability Improvement

Theorem (Y.-Wang)

Suppose A satisfies both T-NSP(t^1, L^1, γ^1) and T-NSP(t^2, L^2, γ^2) where $t^2 < t^1$ and γ^1 and γ^2 are minimal. Then,

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- Result is in dependent of support detectors.

Results for Random Sampling

Theorem (Y.-Wang, an extension to Candés-Tao and Zhang)

For Gaussian random A (or any rank-m matrix A such that $BA^\top = 0$ where $B \in \mathbb{R}^{(n-m)\times m}$ is Gaussian random), a sufficient condition for exact recovery ***with high probability*** *is*

$$\|x_T\|_0 < \frac{C^2}{4} \frac{m-d}{1+\log\frac{n-d}{m-d}},$$

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Application: Bound C and show that

$$-1 < \frac{\partial RHS}{\partial d} < 0,$$

leaving room for incorrect discoveries.

Outline

Numerical Results

Experiment 1: noiseless measurements

- $n = 100$, $m = 50$
- $k = 9, \ldots, 21$. Each k had 200 trials.
- x: sparse Gaussian signals
- A: Gaussian random
- Successful recovery declared if $\|x^{(j)} - x\|_\infty \leq 10^{-3}$
- Thresholds: $\epsilon = \|x^{(j)}\|_\infty / 2^j$

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Empirical exact recovery conditions:

- Basis pursuit:

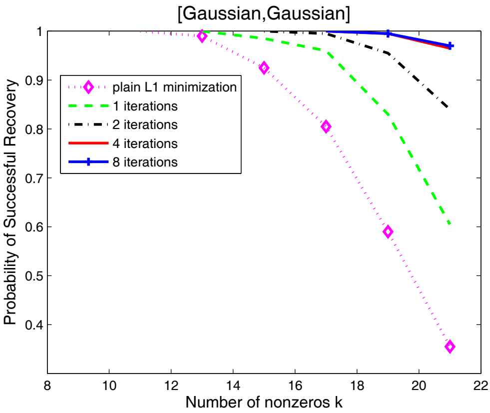
$$k \le \frac{m}{5}$$

- With iterative support detection:

$$k \le \frac{m}{3}$$

Numerical Results

Percentage of Successful Recoveries

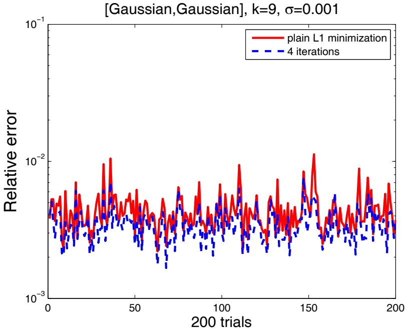


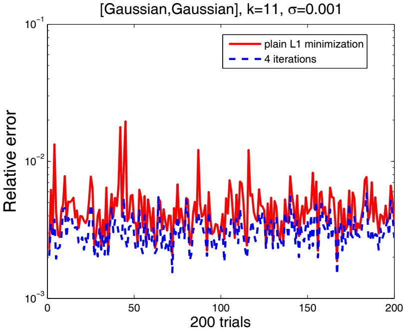
Numerical Results

Experiment 2: noisy measurements

- $n = 100$, $m = 50$
- $k = 9, 11, 15, 19$. Each k had 200 trials.
- x: sparse Gaussian signals
- A: Gaussian random
- $b = Ax + z$, where $z \sim N(0, 0.001)$
- Logarithms of relative errors of $x^{(j)}$ to x are plotted
- Thresholds: $\epsilon = \|x^{(j)}\|_\infty / 2^j$

Numerical Results





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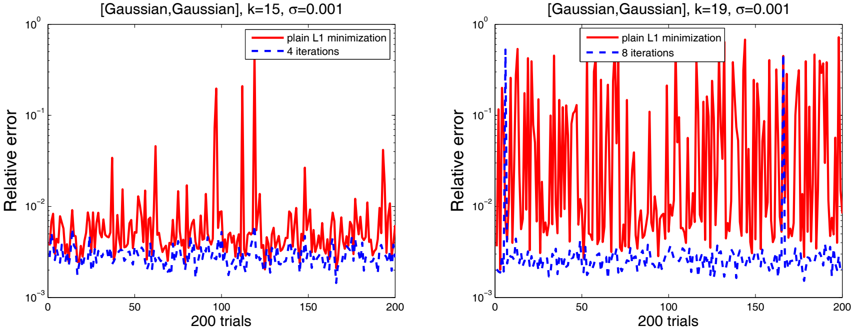
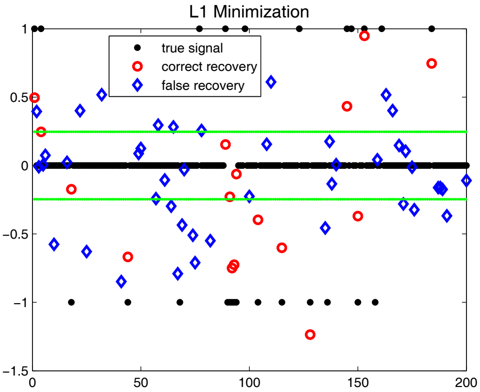


Numerical Results

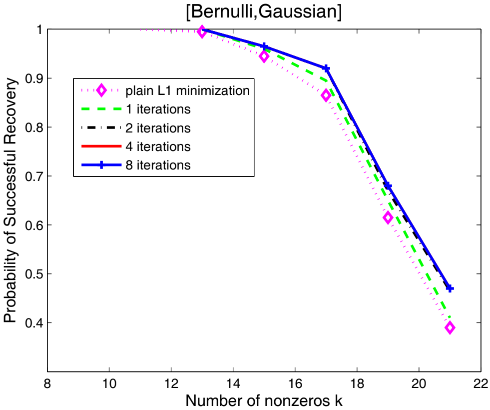
Experiment 3: sparse signals with nonzero $= \pm 1$, noiseless measurements



Excessive false detections!

Numerical Results

Experiment 3: signals with Bernoulli nonzeros, noiseless measurements



Little improvement over basis pursuit.

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- Effective support detection improves CS recovery
- In particular, iterative thresholding is effective on sparse signals with fast decaying distribution of nonzero values
- Computationally tractable
  - one ℓ_1-minimization per iteration, can be warm-started
  - only a small number of iterations are needed

On-going Work

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- other priors: $\|\Phi x\|_p$, $p \leq 1$, and $TV(x)$

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- apply to greedy algorithms (OMP, ROMP, CoSaMP, ...).

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CS Resources: `www.dsp.ece.rice.edu/cs`

Our algorithms: `www.caam.rice.edu/~optimization/L1`

Thank You!