Resolution and robustness of array imaging algorithms

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Compressed Sensing Workshop at Duke

February 26, 2009

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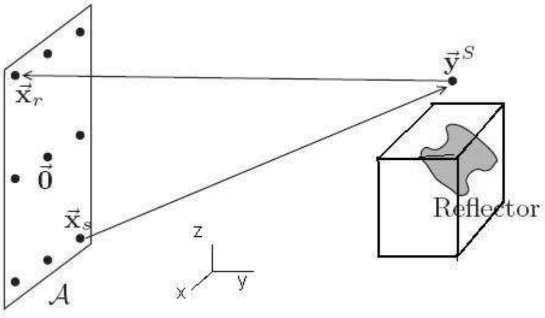
A comparative study and analysis

1. How do several imaging methods perform regarding **resolution** and **robustness**, which means overall stability of the image with some loss of resolution when the signal to noise ratio at the array is low, or when the environment between the target and the sensors is inhomogeneous (random).
2. Imaging methods compared: (i) Basic travel time migration imaging (KM), (ii) Full least squares imaging (LSQ), (iii) MUSIC (multiple signal classification) a subspace projection approach, SVD based, (iv) Direct subspace projection and migration using SVD, (v) Compressed sensing, which means full L^1 minimization with and without subsampling of the array data.

Expected overall performance

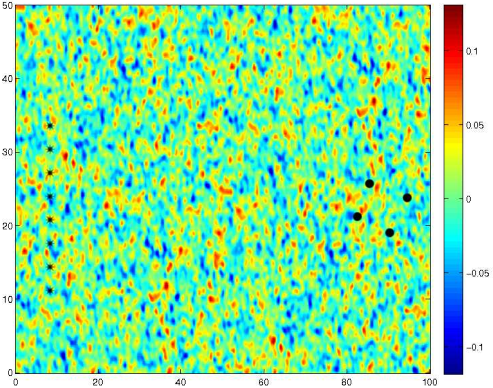
1. KM: Simple to implement for large arrays, low resolution, wide area imaging, robust to additive noise and random media
2. LSQ: More costly to implement for large arrays, improves resolution, localized or wide area imaging, not so robust to noise and random media
3. MUSIC: Simple to implement for small (sparse) targets using the SVD, has good resolution, relatively robust to noise and random media
4. SVD: Simple to implement for small or sparse targets, flexible, relatively robust to noise and random media, can be the basis of more elaborate imaging algorithms to achieve various effects
5. CS: Very effective for sparse targets in homogeneous media, fast algorithm, excellent accuracy even with substantial subsampling. Not robust to noise or random media.

Array imaging schematic



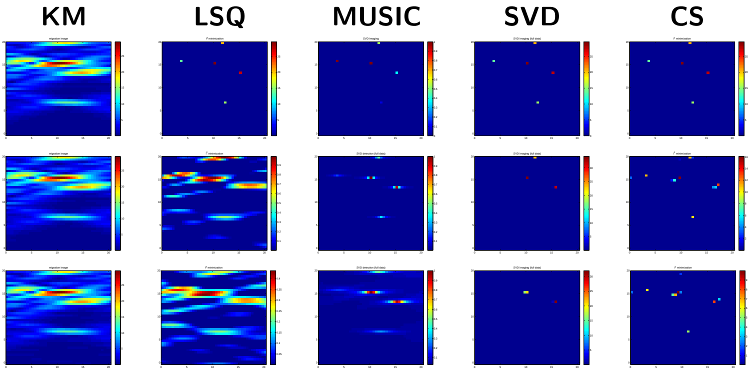
Single frequency imaging. In numerical simulations: Range is 100λ. Array size $100 - 200\lambda$ with half wavelength sensor spacing. Image box: $20\lambda \times 20\lambda$ with half wavelength resolution.

Random medium



Special, phase only (travel time) random perturbations. Typical realization of a random medium with $\sigma_0 = 3\%$ and a correlation length of half a wavelength.

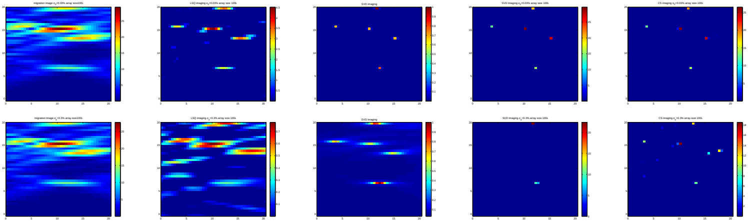
Imaging with additive noise



Top row, homogeneous medium. Middle row, 10 dB additive noise. Bottom row, 0 dB additive noise.

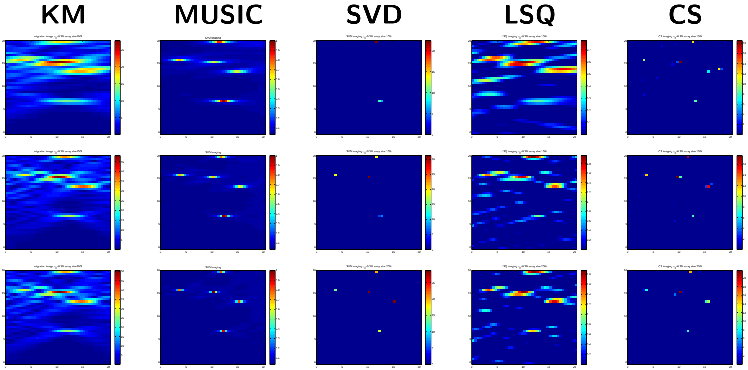
Imaging in a random medium

KM **LSQ** **MUSIC** **SVD** **CS**



Top row, fluctuations at 0.03%. Bottom row, fluctuations at 0.3%.

Random medium and large arrays



The resolution of **KM**, **MUSIC**, **SVD** improves as the size of array increase from 100λ (top) to 150λ (middle) to 200λ (bottom). Fluctuations at 0.3%. Statistical stability phenomenon seen clearly.

Analytical formulation

In a homogeneous medium with wave speed c_0 the time-harmonic Green's function is

$$\widehat{G}_0(\mathbf{x},\mathbf{y},\omega) = \frac{e^{i\frac{\omega}{c_0}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}.$$

The response at $\mathbf{x}_r$ due to signal emitted from $\mathbf{x}_s$ and reflected by scatters (reflectivity ρ_j) is ((s,r) entry of impulse response matrix of array)

$$\widehat{\Pi}(\mathbf{x}_r,\mathbf{x}_s,\omega) = \sum_{j=1}^{M} \rho_j \widehat{G}_0(\mathbf{x}_r,\mathbf{y}_j,\omega)\widehat{G}_0(\mathbf{x}_s,\mathbf{y}_j,\omega).$$

With excitation $\mathbf{f}$ and $\widehat{\mathbf{g}}_0(\mathbf{y},\omega) = [\widehat{G}_0(\mathbf{x}_r,\mathbf{y},\omega)]$, the reflectivity satisfies

$$\mathcal{A}_{\mathbf{f}}\rho = [(\widehat{\mathbf{g}}_0^T(\mathbf{y}_j,\omega)\mathbf{f})\widehat{\mathbf{g}}_0(\mathbf{y}_j,\omega)][\rho_j] = \widehat{\Pi}\mathbf{f} \tag{1}$$

Imaging

Imaging now means solving (**??**) with additional conditions on ρ.

- Kirchhoff Migration (**KM**): $\rho_{KM} = \mathcal{A}_f^H \widehat{\Pi}\mathbf{f}$
- Least squares (**LSQ**): $\min \|\rho\|_2 \quad s.t.\ \mathcal{A}_f \rho = \widehat{\Pi}\mathbf{f}$
- **SVD** method: locate scatters using **MUSIC** via singular value decomposition of $\Pi(\omega)$, then get least square solution with nonnegative constraints to $\tilde{\mathcal{A}}_f \rho = \widehat{\Pi}\mathbf{f}$ where $\tilde{\mathcal{A}}_f$ is formed by columns of $\mathcal{A}_f$ corresponding to large value of **MUSIC** functional.
- Compressed sensing (**CS**): $\min \|\rho\|_1 \quad s.t.\ \mathcal{A}_f \rho = \widehat{\Pi}\mathbf{f}$

Additive noise

Noise $\mathbf{e}$ causes perturbations of the response matrix

$$\mathcal{A}_{\mathfrak{f}}\rho = (\widehat{\Pi} + \Delta\widehat{\Pi})\mathbf{f} := \widehat{\Pi}\mathbf{f} + \mathbf{e}$$

Assume that the noise level $\|\mathbf{e}\|_2 \leqslant \varepsilon$ can be estimated. Replace with inequality constraint $\|\mathcal{A}_{\mathfrak{f}}\rho - \widehat{\Pi}\mathbf{f}\|_2 \leqslant \varepsilon$ in **LSQ** and **CS**. **KM** and **SVD** do not change.

Random medium

We model wave speed in a random medium by

$$c^{-2} = c_0^{-2}(1 + \sigma_0 \mu)$$

where $\mu(\cdot)$ is zero mean, isotropic, weakly stationary random fluctuation. A realization of such random fluctuation field was shown above. We model the Green function in random medium by

$$\widehat{G}(\mathbf{x}, \mathbf{y}, \omega) = \widehat{G}_0(\mathbf{x}, \mathbf{y}, \omega) e^{i \frac{\omega \sigma_0}{c_0} |\mathbf{x} - \mathbf{y}| \int_0^1 \mu\left(\frac{\mathbf{x}}{\ell} + \frac{s}{\ell}(\mathbf{y} - \mathbf{x})\right) ds}$$

and the response matrix is obtained by substituting $\widehat{G}_0$ by $\widehat{G}$.
As in the case of additive noise, we use inequality constraint in **LSQ** and **CS**. However, the noise level ε is usually impossible to estimate accurately, which limits the use of minimization approaches in random medium.

Moment formula; effective aperture

KM and **SVD** satisfy *Statistical Stability*: Better resolution can be obtained by increasing the size of array.

Result 1. For random medium model we have:

$$\mathbb{E}\left(\widehat{G}(\mathbf{x},\mathbf{y},\omega)\overline{\widehat{G}(\mathbf{x},\mathbf{y}',\omega)}\right) \approx \widehat{G}_0(\mathbf{x},\mathbf{y},\omega)\overline{\widehat{G}_0(\mathbf{x},\mathbf{y}',\omega)}e^{-\frac{\left(\frac{\omega}{c_0}\right)^2|\mathbf{y}-\mathbf{y}'|^2}{2L^2}a_e^2},$$

where

$$a_e = \sigma_0 L\left(-R(0) - \frac{2}{3}\frac{L}{\ell}\int_0^\infty dt\frac{R'(t)}{t}\right)^{1/2}$$

is the effective aperture of random medium.

Here $R(\cdot)$ is the correlation function of the random fluctuations.

Statistical stability

Result 2. As size of array becomes larger, we also have

$$\mathbb{E}\left|\widehat{G}(\mathbf{x},\mathbf{y},\omega)\overline{\widehat{G}(\mathbf{x},\mathbf{y}',\omega)} - \mathbb{E}\left(\widehat{G}(\mathbf{x},\mathbf{y},\omega)\overline{\widehat{G}(\mathbf{x},\mathbf{y}',\omega)}\right)\right|^2$$
$$\approx \frac{1}{(4\pi)^4|\mathbf{x}-\mathbf{y}|^2|\mathbf{x}-\mathbf{y}'|^2}\left(1 - e^{-\left(\frac{\omega}{c_0}\right)^2\frac{a_e^2}{L^2}|\mathbf{y}-\mathbf{y}'|^2}\right),$$

and therefore

$$\frac{\mathbb{E}\left|\widehat{g}^H(\mathbf{y},\omega)\widehat{g}(\mathbf{y}',\omega) - \mathbb{E}\left(\widehat{g}^H(\mathbf{y},\omega)\widehat{g}(\mathbf{y}',\omega)\right)\right|^2}{\mathbb{E}\left|\widehat{g}(\mathbf{y},\omega)\right|^2} \to 0, \qquad a \to \infty.$$

Summary and current projects

- Optimization based imaging (LSQ or CS) is very sensitive to noise and random media
- Broadband imaging helps (frequency diversity) but does not address the real issue
- Interferometric imaging (broadband mostly) does address the issues and can be very effective **with optimization in subspace selection and illumination** (BPT Journal of the Acoustical Society of America, vol 122 (2007), pp. 3507-3518)
- Interferometric methods with L^1 optimization have not been explored up to now
- Imaging moving targets in heavy clutter is a good application for the evolving methodology