Model-based Compressive Sensing

Richard Baraniuk

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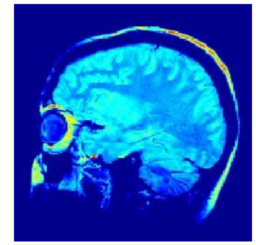
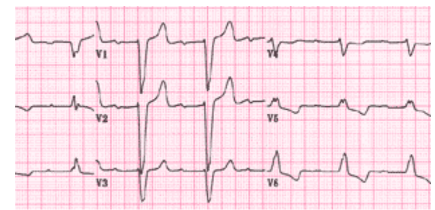
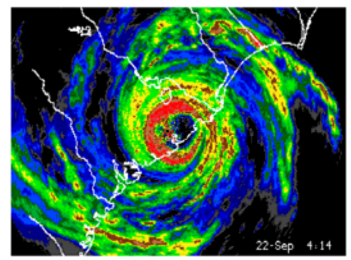
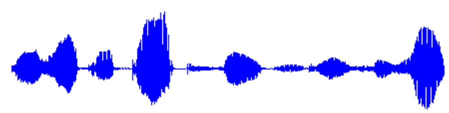
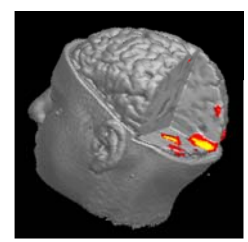
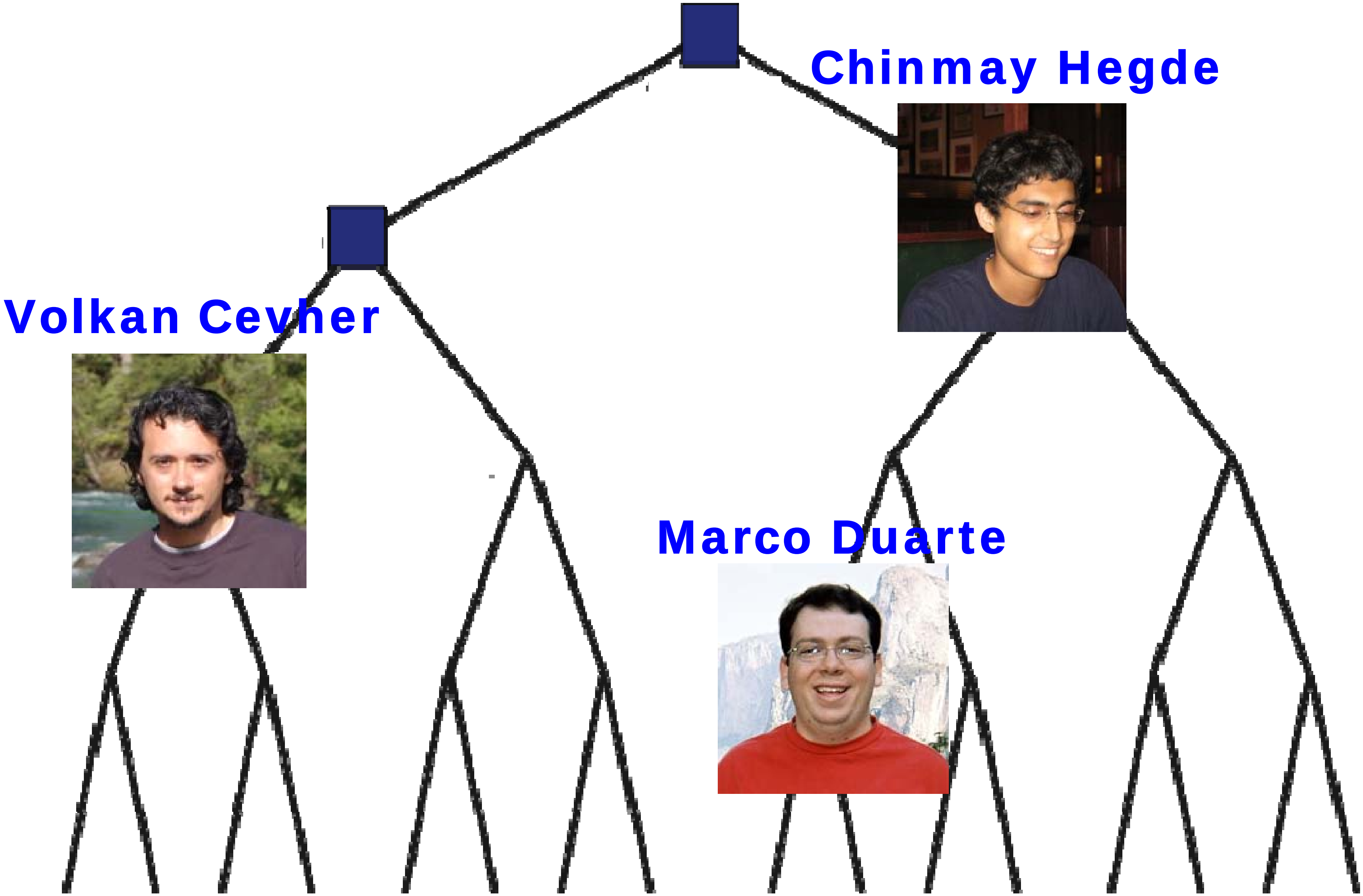






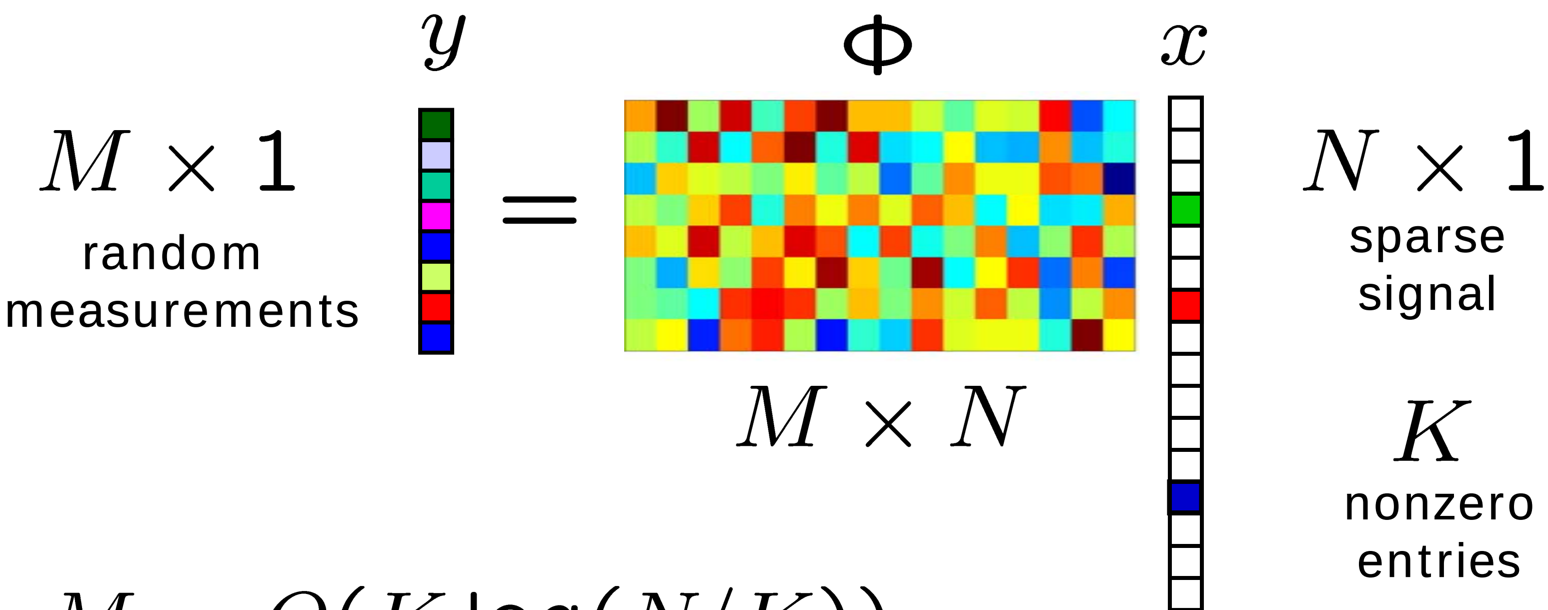





Chinmay Hegde
Volkan Cevher
Marco Duarte

Compressive Sensing

- **Sensing** via randomized dimensionality reduction



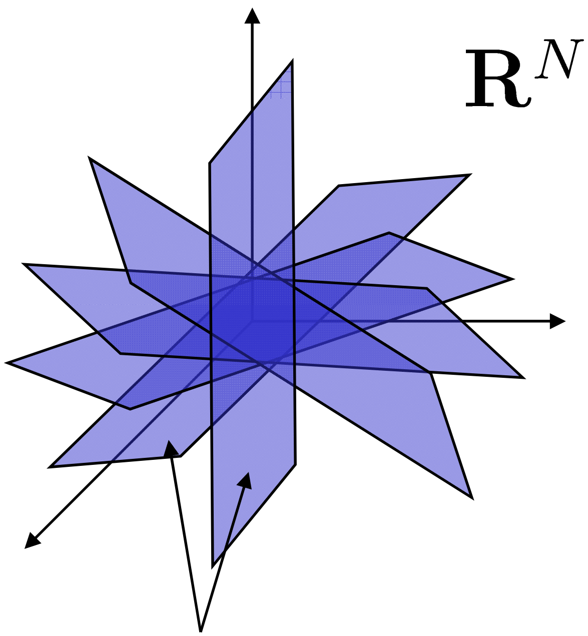
$$M = O(K \log(N/K))$$

- **Recovery:** solve an ill-posed inverse problem

  exploit the geometrical structure of sparse/compressible signals

Restricted Isometry Property (RIP)

- Preserve the structure of sparse/compressible signals

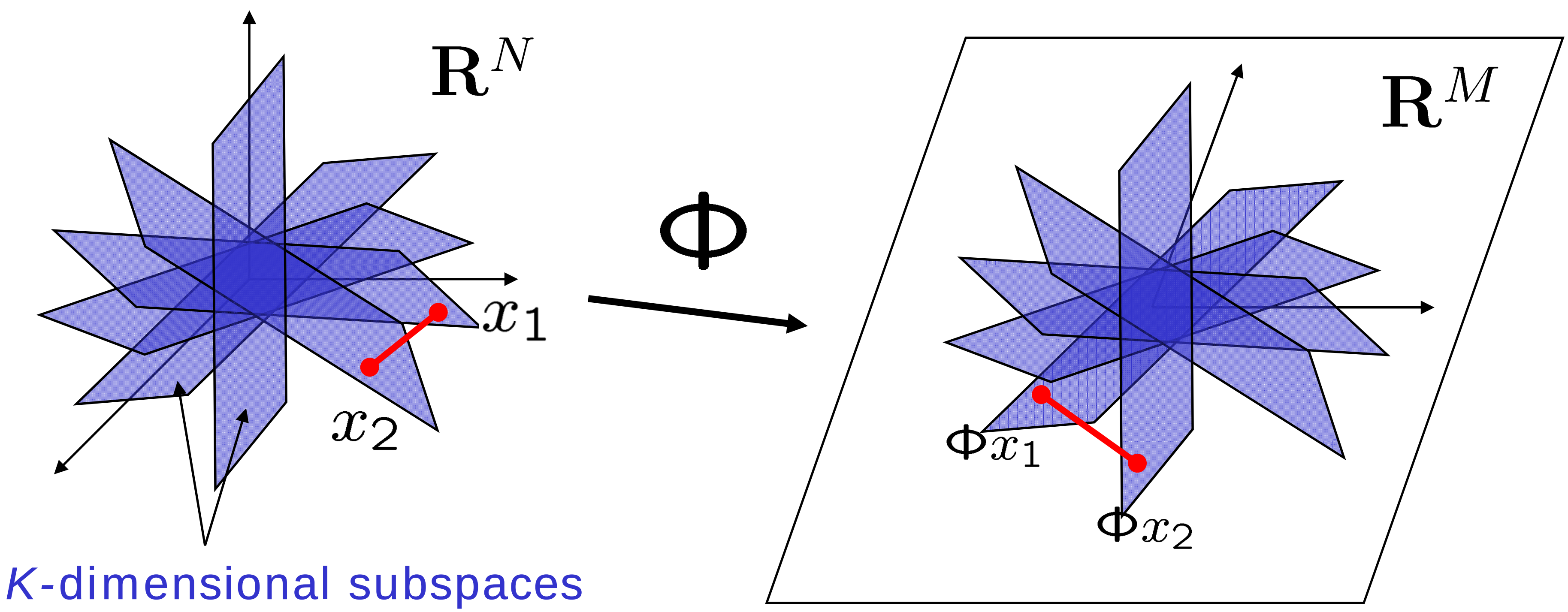


K-dimensional subspaces

Restricted Isometry Property (RIP)

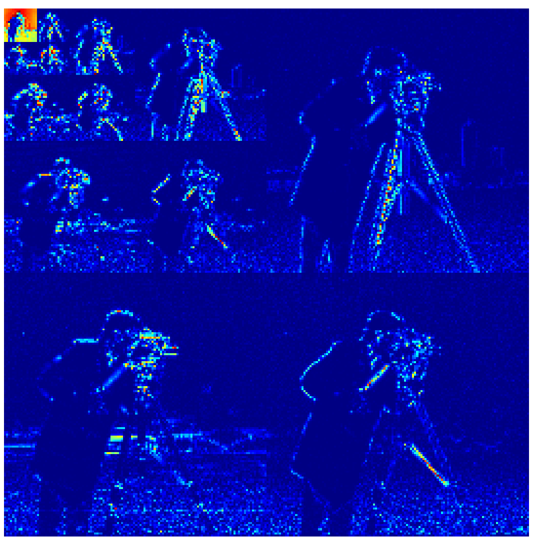
- Preserve the structure of sparse/compressible signals
- RIP of order *2K* implies: for all *K*-sparse x_1 and x_2

$$(1-\delta_{2K}) \leq \frac{\|\Phi x_1 - \Phi x_2\|_2^2}{\|x_1 - x_2\|_2^2} \leq (1+\delta_{2K})$$

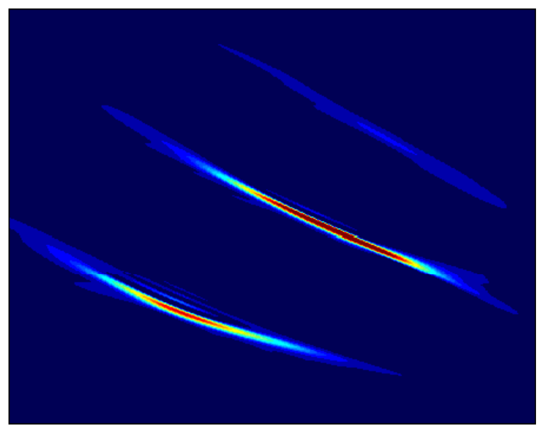


Beyond Sparse Models

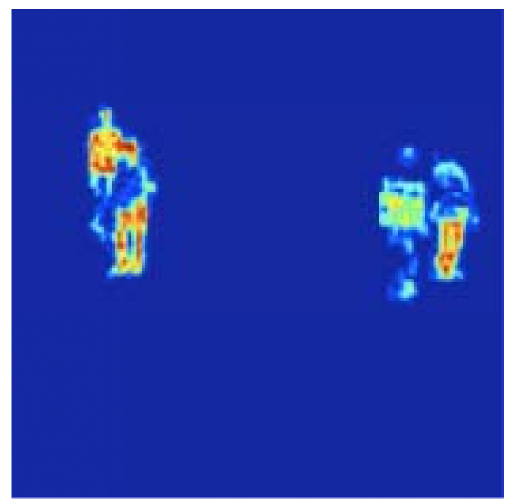
- Sparse/compressible signal model captures **simplistic primary structure**



wavelets:
natural images



Gabor atoms:
chirps/tones

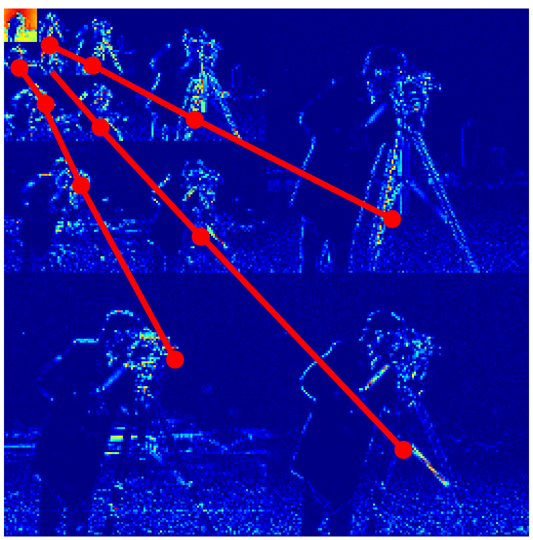


pixels:
background subtracted images

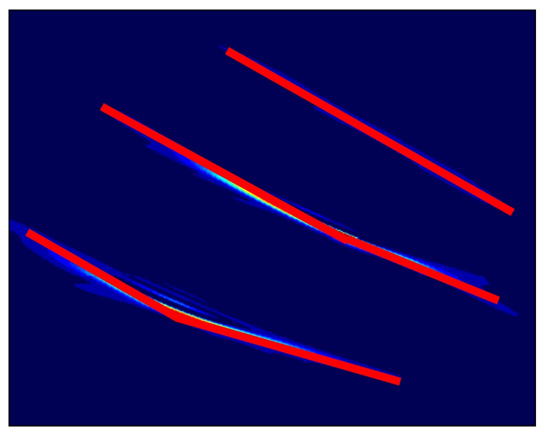
Beyond Sparse Models

- Sparse/compressible signal model captures **simplistic primary structure**

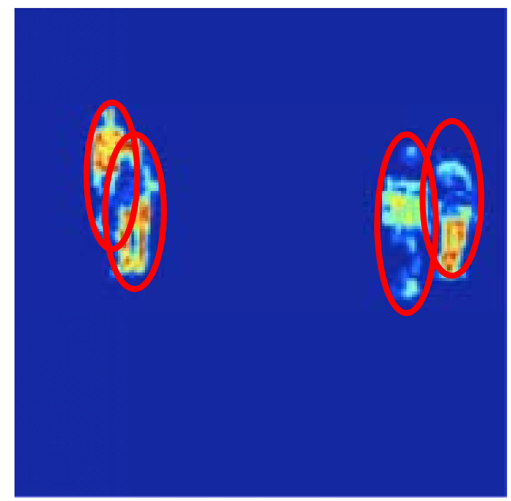
- Modern compression/processing algorithms capture **richer secondary coefficient structure**



wavelets:
natural images



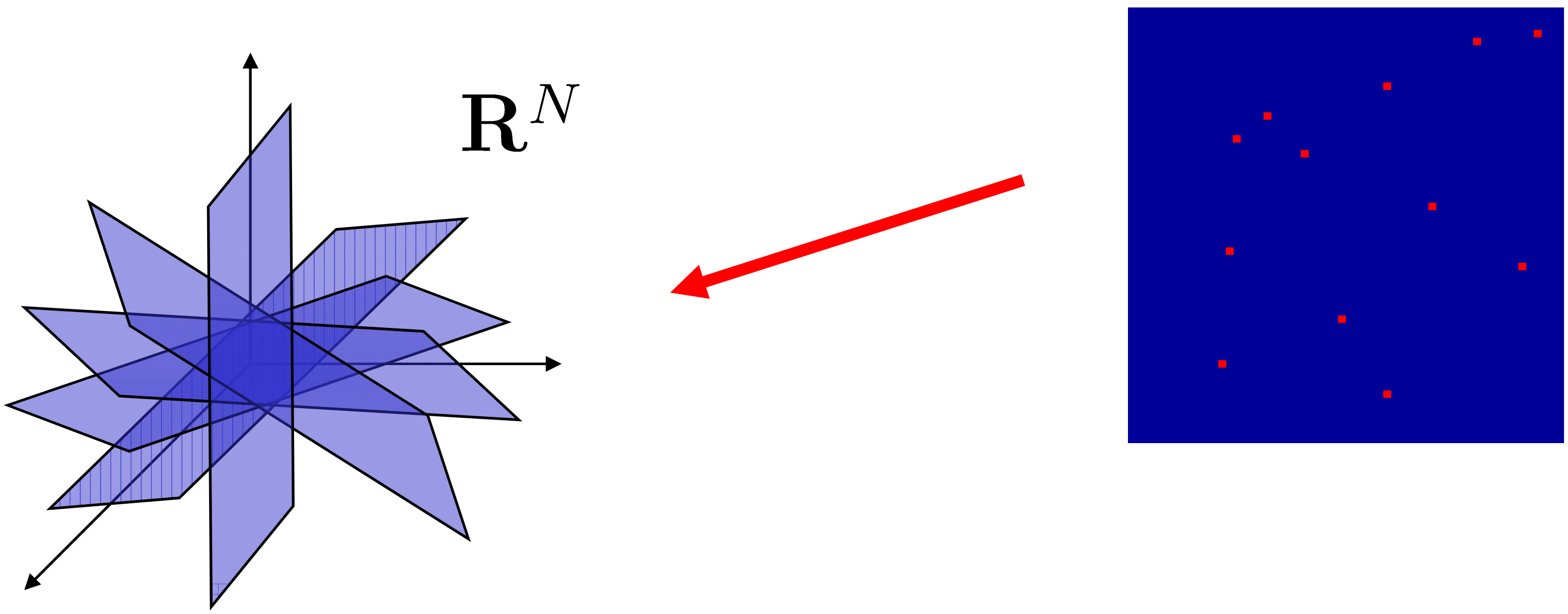
Gabor atoms:
chirps/tones



pixels:
background subtracted
images

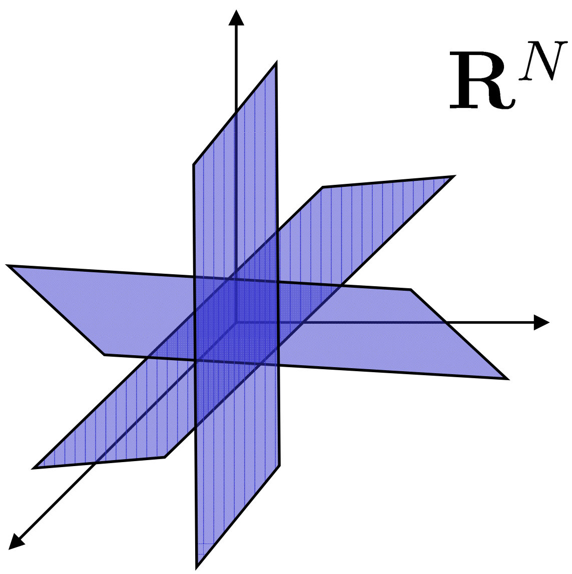
Sparse Signals

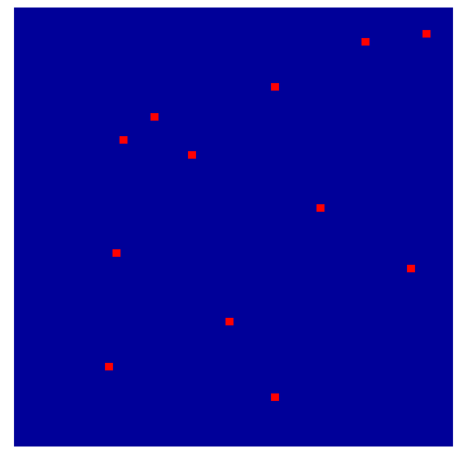
- Defn: ***K*-sparse signals** comprise a particular set of *K*-dim canonical subspaces

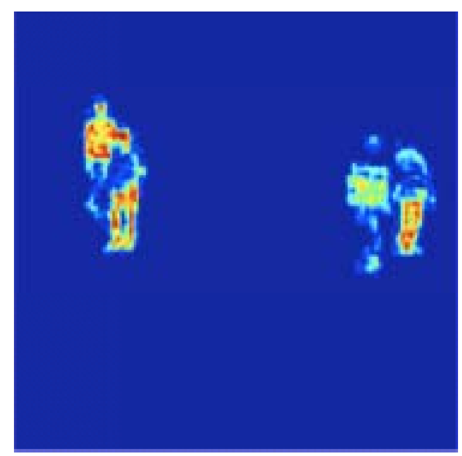


Model-Sparse Signals

- Defn: A ***K*-sparse signal *model*** comprises a particular (*reduced*) set of *K*-dim canonical subspaces [Blumensath and Davies]





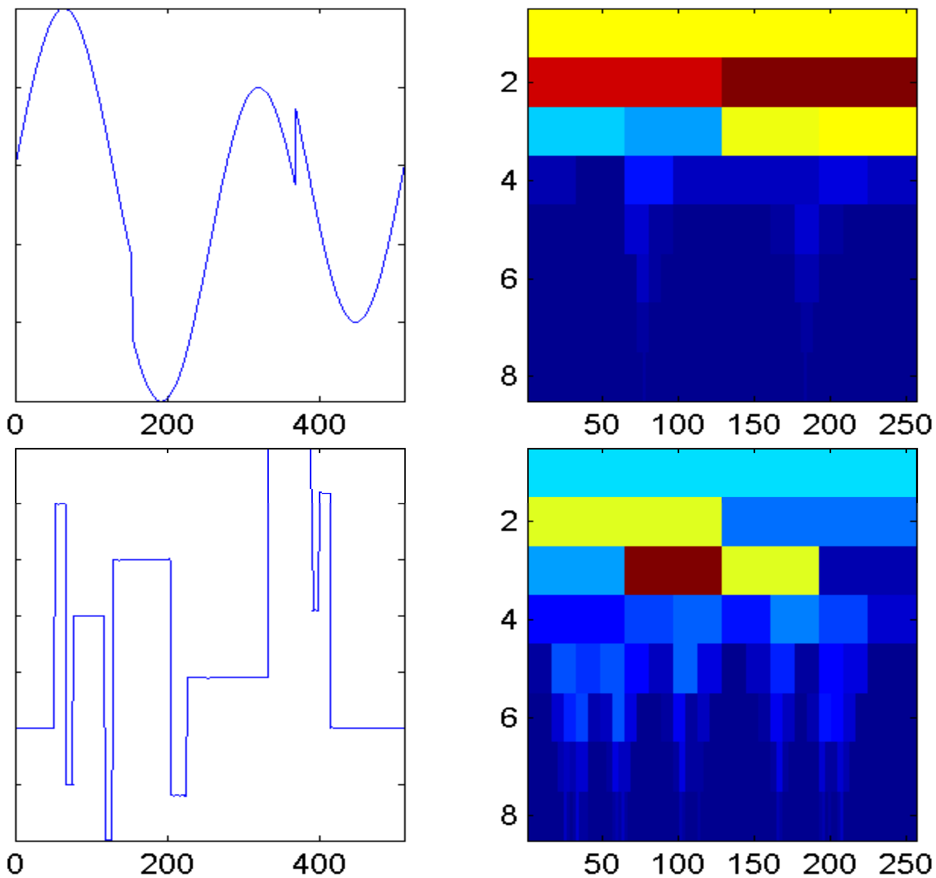


- Fewer subspaces
  < > relaxed RIP
  < > stable recovery using
  fewer measurements *M*

Model-based CS

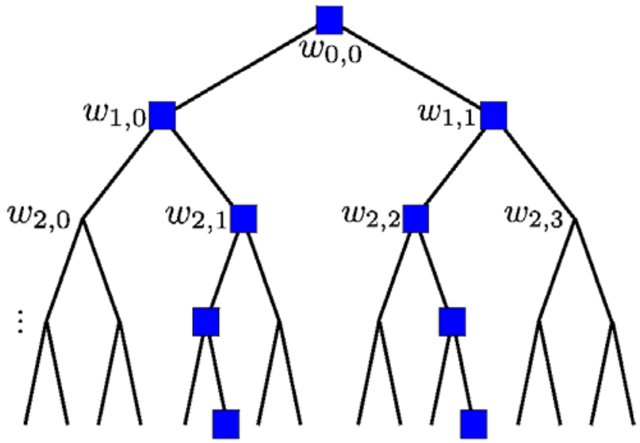
Running Example:
Tree-Sparse Signals

Wavelet Sparse

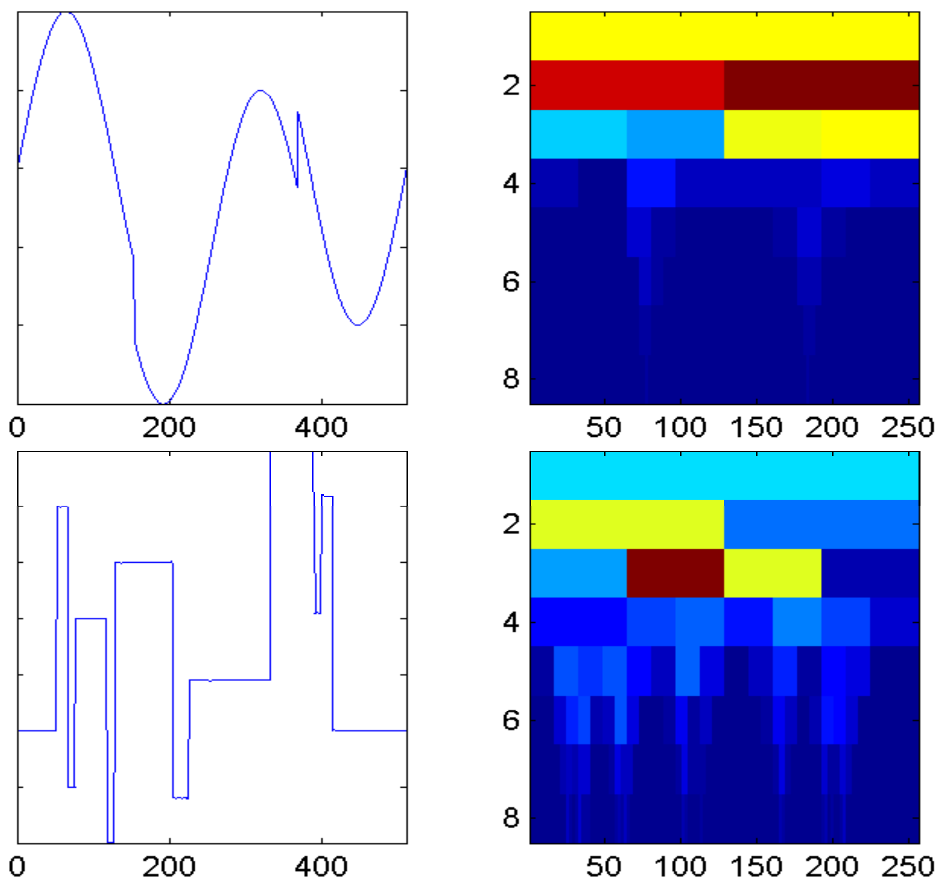


- Typical of wavelet transforms
  of natural signals
  and images
  (piecewise smooth)

Tree-Sparse

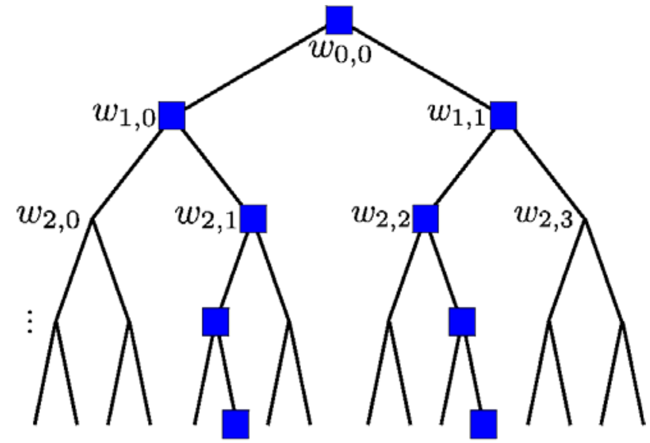


- **Model:** *K*-sparse coefficients
  + significant coefficients
  lie on a **rooted subtree**



- Typical of wavelet transforms of natural signals and images (piecewise smooth)

Tree-Sparse

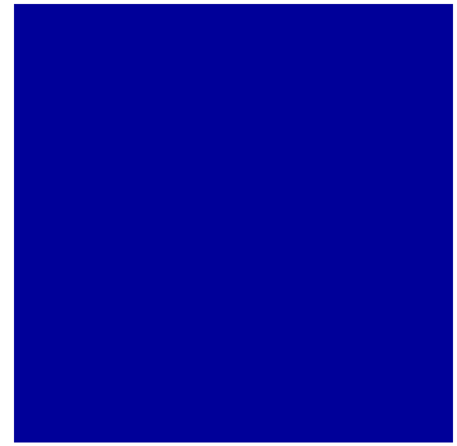


- **Model:** *K*-sparse coefficients
  + significant coefficients lie on a rooted subtree

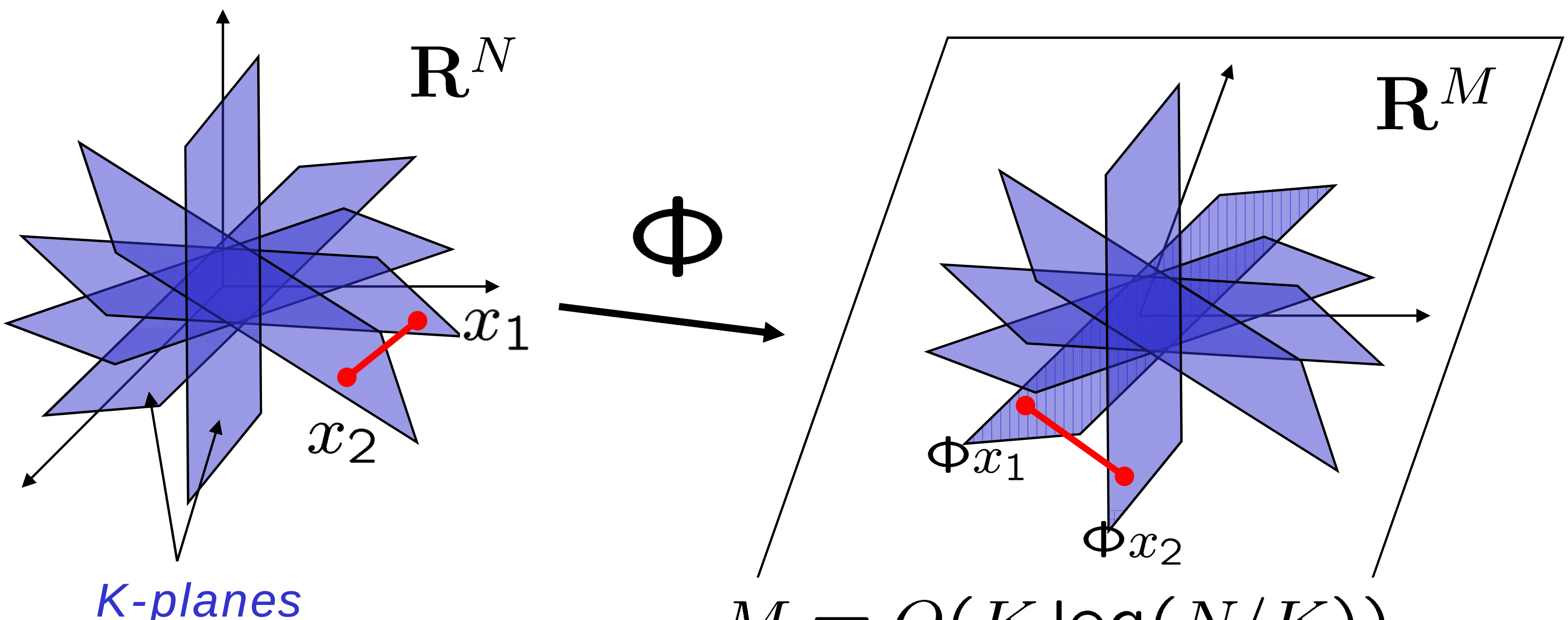
- **Sparse approx:** find best set of coefficients
  - sorting
  - hard thresholding

- **Tree-sparse approx:** find best rooted subtree of coefficients
  - CSSA [B]
  - dynamic programming [Donoho]

Wavelet Sparse



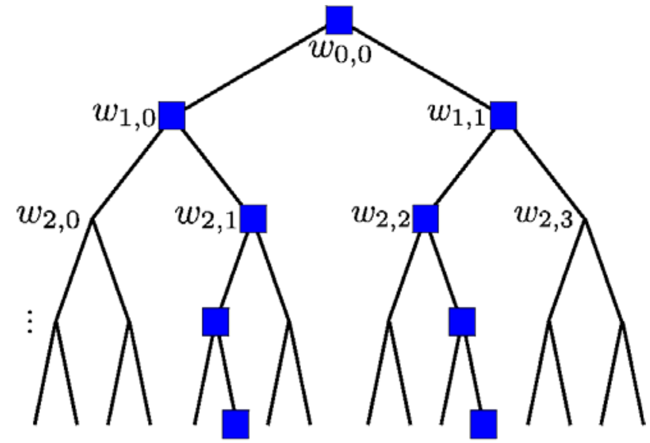
- **RIP:** stable embedding



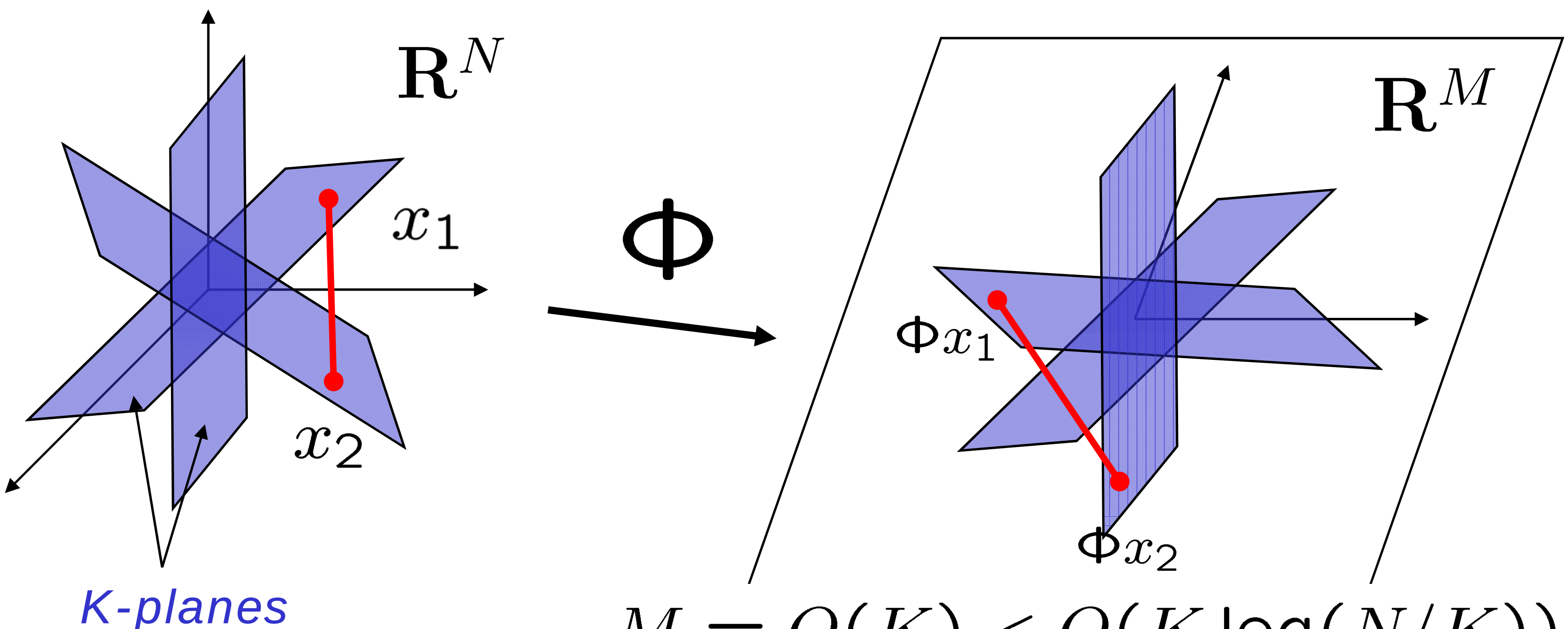
$$M = O(K \log(N/K))$$

Tree-Sparse

- **Model:** K-sparse coefficients
  + significant coefficients
  lie on a rooted subtree

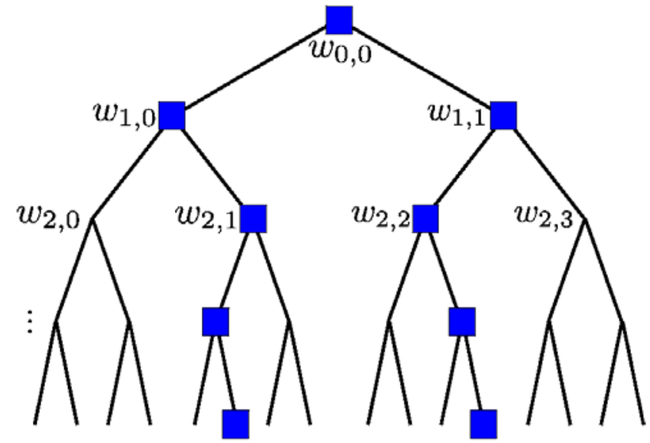


- **Tree-RIP:** stable embedding
  [Blumensath and Davies]



$$M = O(K) < O(K \log(N/K))$$

Tree-Sparse



- **Model:** *K*-sparse coefficients
  + significant coefficients
  lie on a rooted subtree

- **Tree-RIP:** stable embedding
  [Blumensath and Davies]

- **Recovery:** inject tree-sparse approx into
  IHT/CoSaMP

Recall: Iterated Thresholding

goal: given $y = \Phi x$, recover a sparse x

initialize: $\widehat{x}_0 = 0$, $r = y$, $i = 0$

iteration:

[Nowak, Figueiredo; Kingsbury, Reeves; Daubechies, Defrise, De Mol; Blumensath, Davies; ...]

- $i \leftarrow i + 1$
- $b \leftarrow \widehat{x}_{i-1} + \Phi^T r$ **update signal estimate**
- $\widehat{x}_i \leftarrow \text{thresh}(b, K)$ **prune signal estimate** (best K-term approx)
- $r \leftarrow y - \Phi \widehat{x}_i$ **update residual**

return: $\widehat{x} \leftarrow \widehat{x}_i$

Iterated **Model** Thresholding

goal: given $y = \Phi x$, recover a sparse x

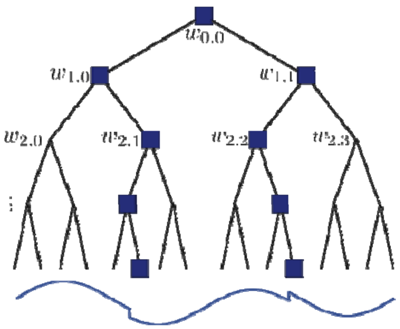
initialize: $\widehat{x}_0 = 0$, $r = y$, $i = 0$

iteration:

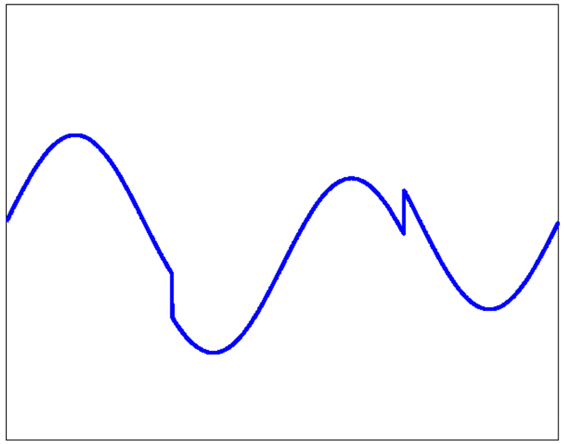
- $i \leftarrow i + 1$
- $b \leftarrow \widehat{x}_{i-1} + \Phi^T r$ — **update signal estimate**
- $\widehat{x}_i \leftarrow \underline{\mathcal{M}(b, K)}$ — **prune signal estimate** (best *K*-term model approx)
- $r \leftarrow y - \Phi \widehat{x}_i$ — **update residual**

return: $\widehat{x} \leftarrow \widehat{x}_i$

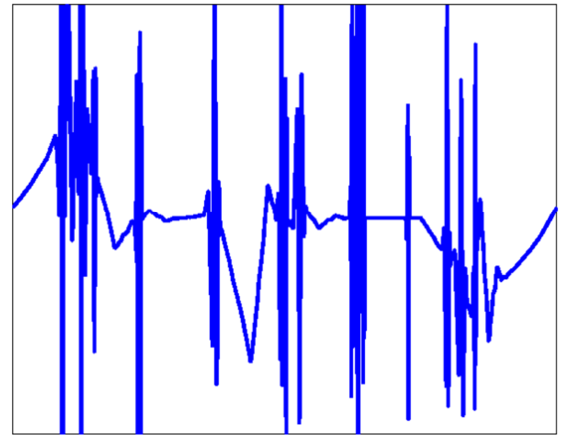
Tree-Sparse Signal Recovery



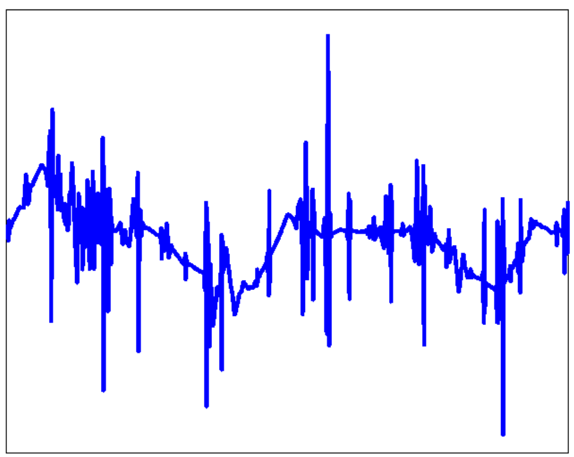
$N = 1024$
$M = 80$



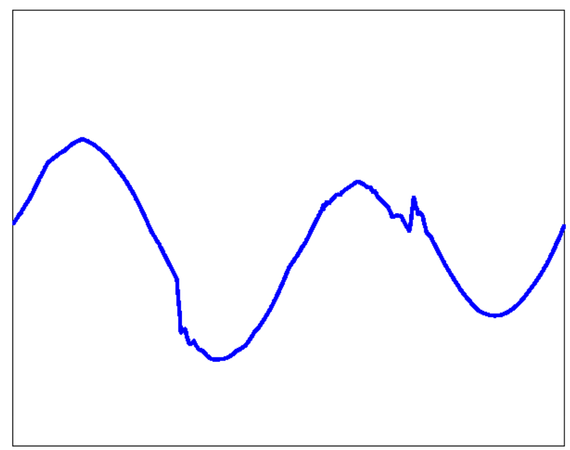
target signal



CoSaMP, (MSE=1.12)



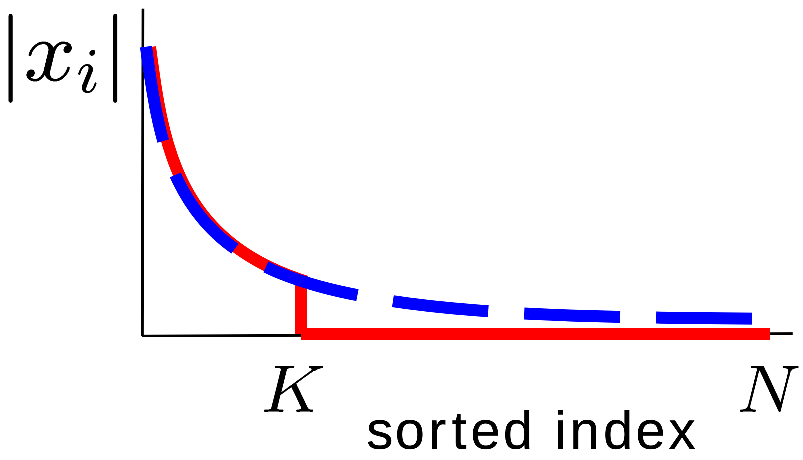
L1-minimization (MSE=0.751)



Tree-sparse CoSaMP (MSE=0.037)

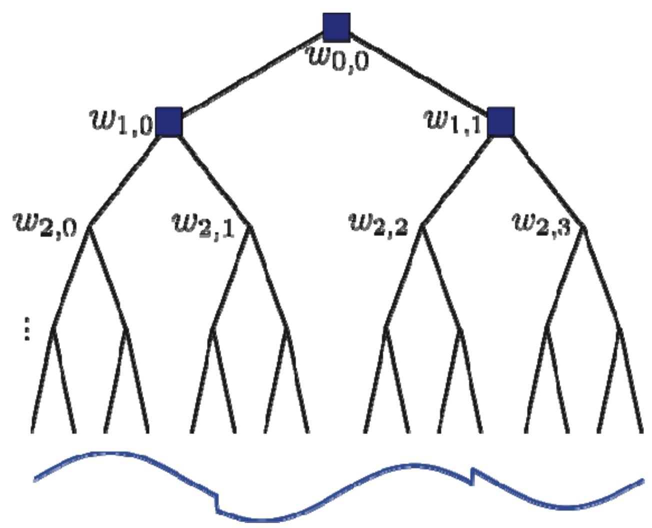
Compressible Signals

- Real-world signals are compressible, not sparse
- **Compressible** < > approximable by sparse
  - compressible signals lie close to a union of subspaces
  - power-law decay in coefficients
  - ie: approximation error decays rapidly as $K \rightarrow \infty$
- If Φ has RIP, then both sparse and compressible signals are stably recoverable via LP or greedy alg



Model-Compressible Signals

- **Model-compressible** <> approximable by model-sparse
  - model-compressible signals lie close to a reduced union of subspaces
  - ie: model-approx error decays rapidly as $K \longrightarrow \infty$



Model-Compressible Signals

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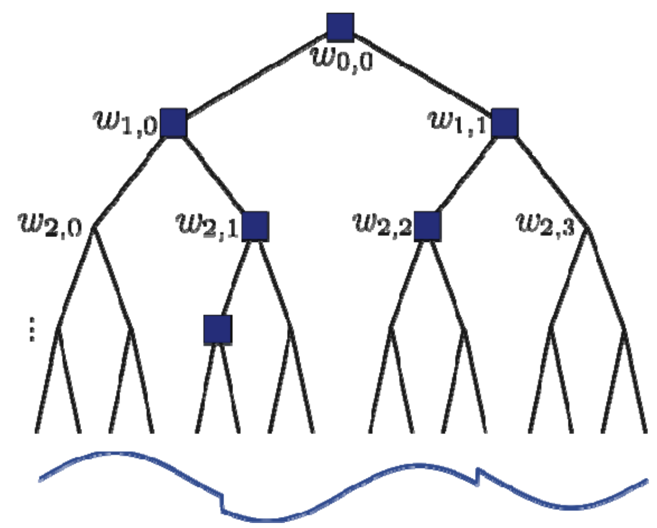

Model-Compressible Signals

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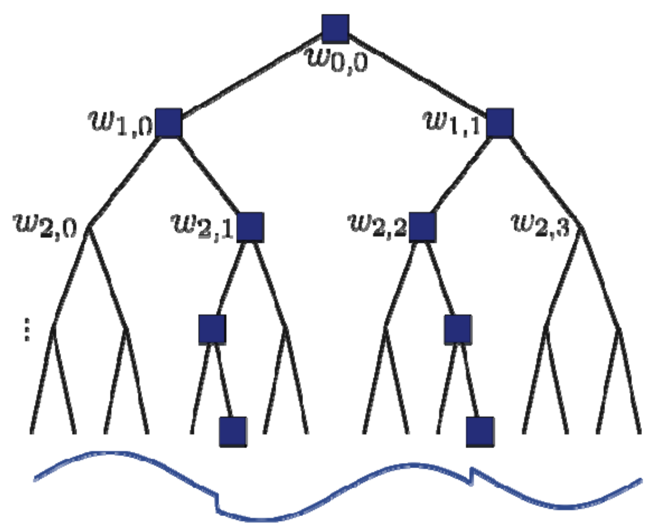
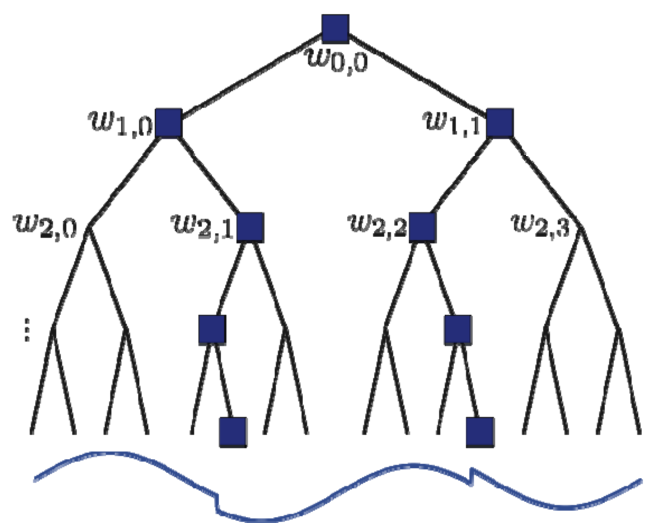


Model-Compressible Signals

- **Model-compressible** <> approximable by model-sparse
  - model-compressible signals lie close to a reduced union of subspaces
  - ie: model-approx error decays rapidly as $K \longrightarrow \infty$

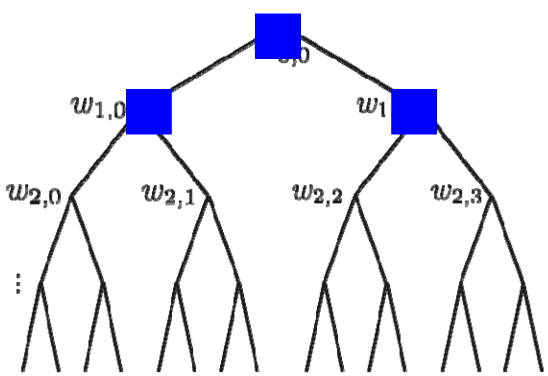
- **New result:** while model-RIP enables stable model-sparse recovery, model-RIP is *not sufficient* for stable model-compressible recovery!



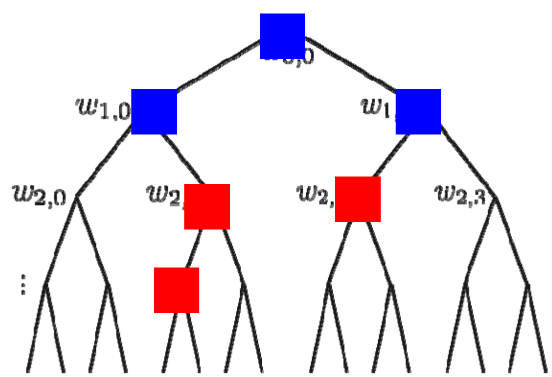
Stable Recovery

- **Result:** Stable model-compressible signal recovery requires that Φ have both:
  - RIP + **Restricted Amplification Property**

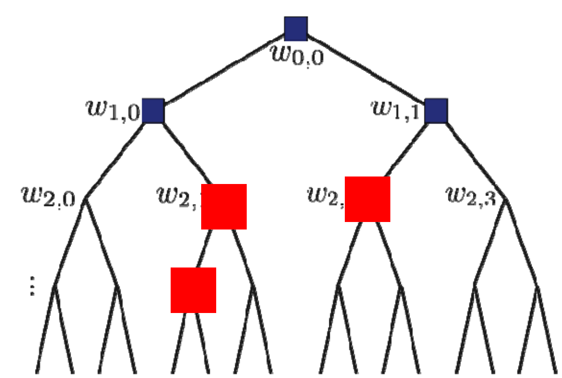
- **RAmP:** controls nonisometry of Φ in the approximation's **residual subspaces**



optimal *K*-term model recovery (error controlled by Φ RIP)



optimal 2*K*-term model recovery (error controlled by Φ RIP)



residual subspace (error *not* controlled by Φ RIP)

Tree-RIP, Tree-RAmP

Theorem: An *M*x*N* iid subgaussian random matrix has the Tree(*K*)-RIP if

$$M \geq \begin{cases} \frac{2}{c\delta^2_{\mathcal{T}_K}}\left(K \ln \frac{48}{\delta_{\mathcal{T}_K}} + \ln \frac{512}{Ke^2} + t\right) & \text{if } K < \log_2 N \\ \frac{2}{c\delta^2_{\mathcal{T}_K}}\left(K \ln \frac{24e}{\delta_{\mathcal{T}_K}} + \ln \frac{2}{K+1} + t\right) & \text{if } K \geq \log_2 N \end{cases}$$

Theorem: An *M*x*N* iid subgaussian random matrix has the Tree(*K*)-RAmP if

$$M \geq \begin{cases} \frac{2}{\left(\sqrt{1+\epsilon_K}-1\right)^2}\left(10K + 2\ln \frac{N}{K(K+1)(2K+1)} + t\right) & \text{if } K \leq \log_2 N \\ \frac{2}{\left(\sqrt{1+\epsilon_K}-1\right)^2}\left(10K + 2\ln \frac{601N}{K^3} + t\right) & \text{if } K > \log_2 N \end{cases}$$

Performance

- Using model-based IHT, CoSaMP with RIP+RAmP

- **Model-sparse signals**
  - noise-free measurements: exact recovery
  - noisy measurements: stable recovery

- **Model-compressible signals**
  - recovery as good as *K*-model-sparse approximation

$$\|x - \widehat{x}\|_{\ell_2} \leq C_1 \|x - x_K\|_{\ell_2} + C_2 \frac{\|x - x_K\|_{\ell_1}}{K^{1/2}} + C_3 \epsilon$$

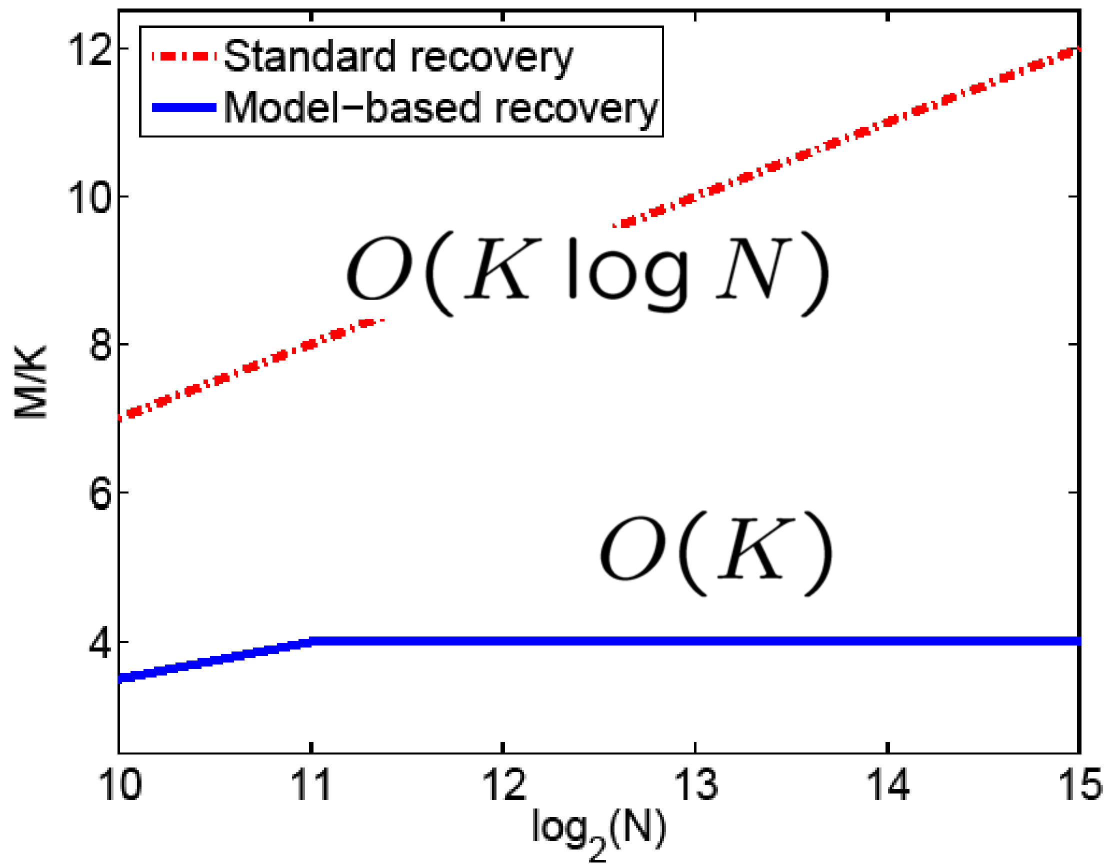
CS recovery error — signal model *K*-term approx error — signal model *K*-term approx error — noise

Simulation

- Number samples for correct recovery

- Piecewise cubic signals + wavelets

- Models/algorithms:
  - sparse (CoSaMP)
  - tree-sparse (tree-CoSaMP)

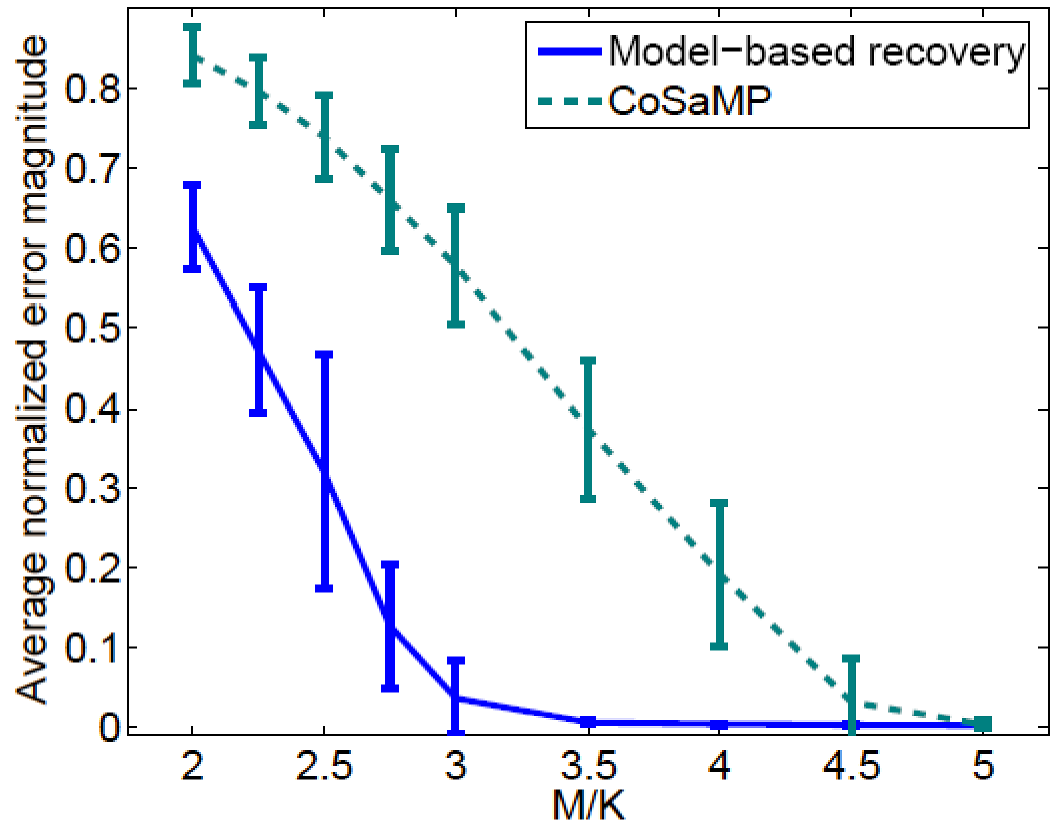


Simulation

- Recovery performance (MSE) vs. number of measurements

- Piecewise cubic signals + wavelets

- Models/algorithms:
  - sparse (CoSaMP)
  - tree-sparse (tree-CoSaMP)



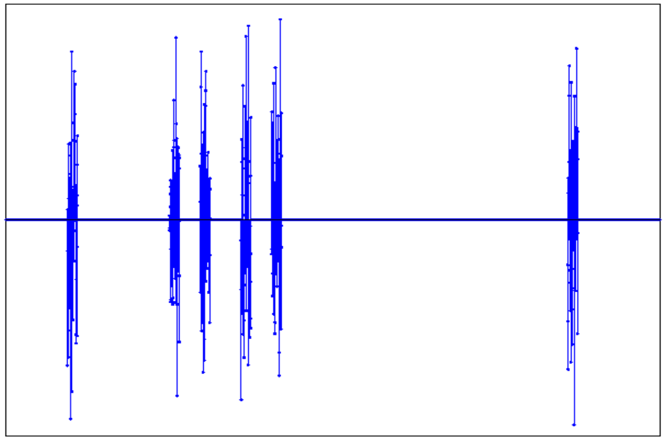
Other Useful Models

- When the model-based framework makes sense:
  - **model** with
    - Nested approximation property (NAP)
    - fast approximation algorithm
  - sensing **matrix** Φ with
    - model-RIP
    - model-RAmP

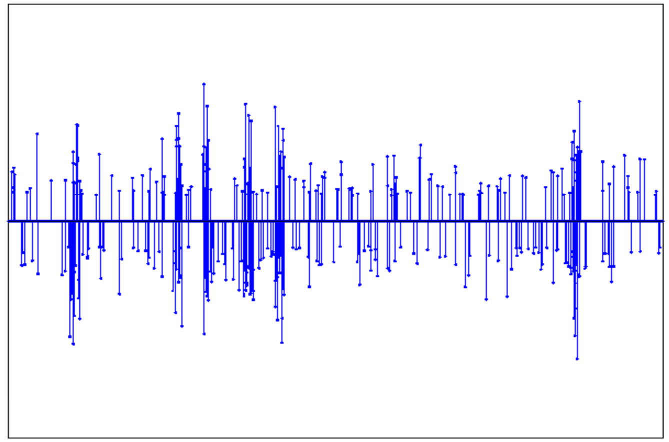
- Ex: block sparsity / signal ensembles
  [Tropp, Gilbert, Strauss], [Stojnic, Parvaresh, Hassibi], [Eldar, Mishali], [Baron, Duarte et al], [B, C, Duarte, Hegde]

- Ex: clustered signals
  [C, Duarte, Hegde, B], [C, Indyk, Hegde, Duarte, B]

Block-Sparse Signal

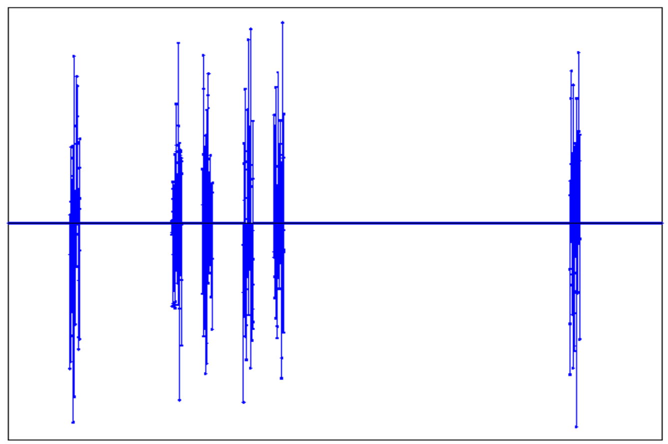


target



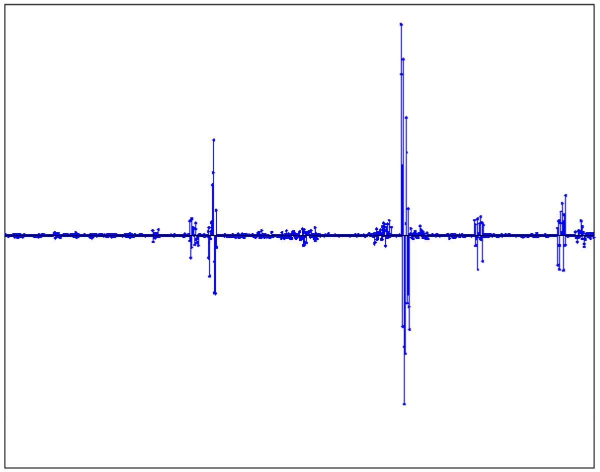
CoSaMP (MSE = 0.723)

N = 4096
K = 6 active blocks
J = block length = 64
M = 2.5JK = 960 msnts

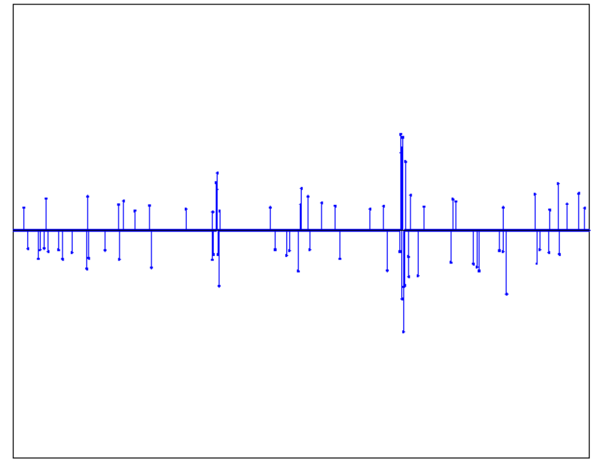


block-sparse model recovery
(MSE=0.015)

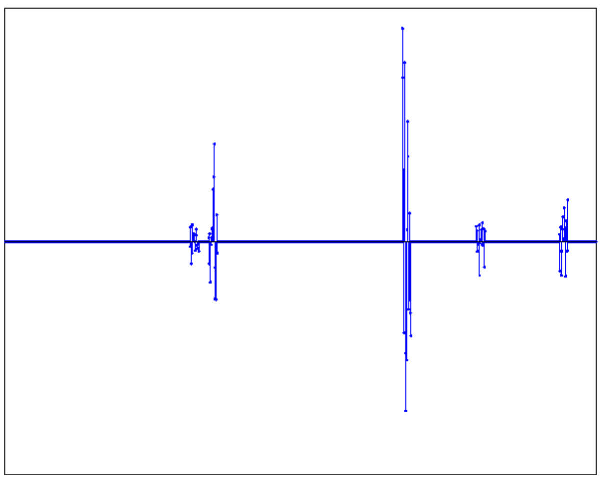
Block-Compressible Signal



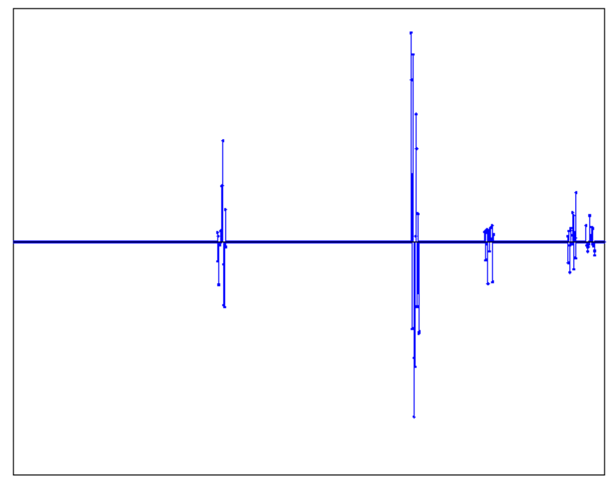
target



CoSaMP (MSE=0.711)



best 5-block approximation
(MSE=0.116)



block-sparse recovery
(MSE=0.195)

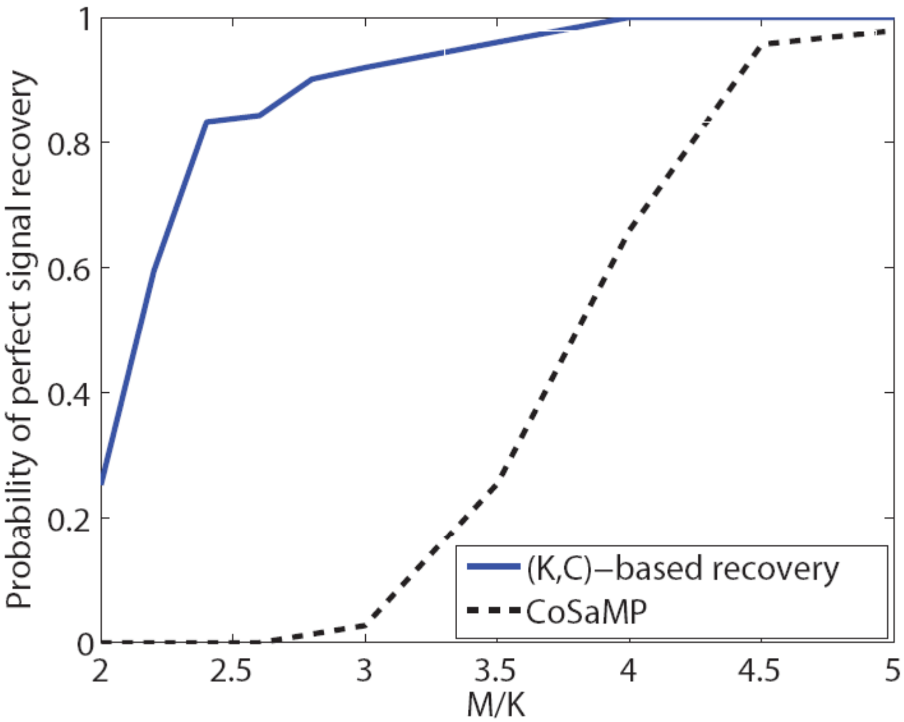
Clustered Sparsity

- ***(K*,*C*) sparse signals** (1-D)
  - *K*-sparse within at most *C* clusters

- For stable recovery (model-RIP + RAmP)

$$M = O(K + C\log(N/C))$$

- Model approximation using **dynamic programming**

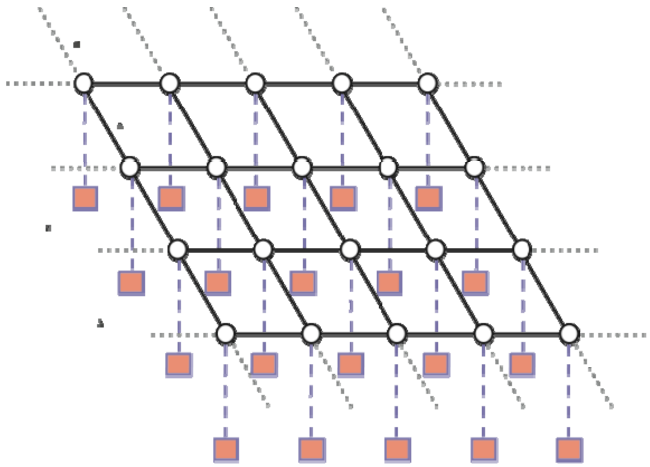


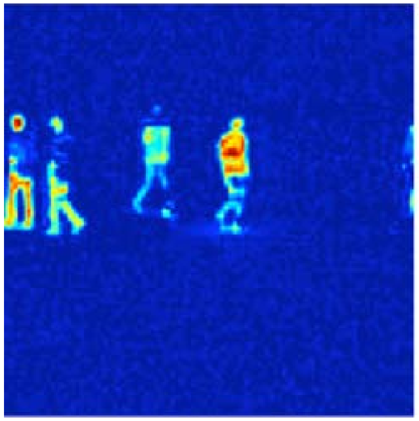
[Cevher, Indyk, Hedge, B; Sampta 2009]

Clustered Sparsity

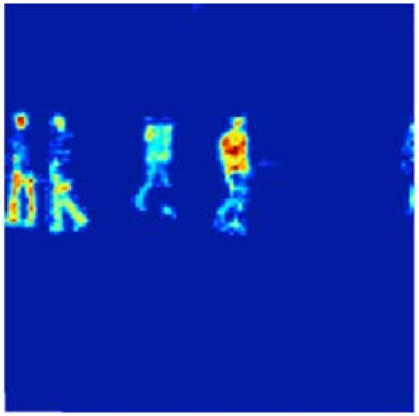
- Model clustering of significant pixels in space domain using **graphical model** (Ising MRF)
- Ising model approximation via **graph cuts**

[Cevher, Duarte, Hedge, B; NIPS 2008]

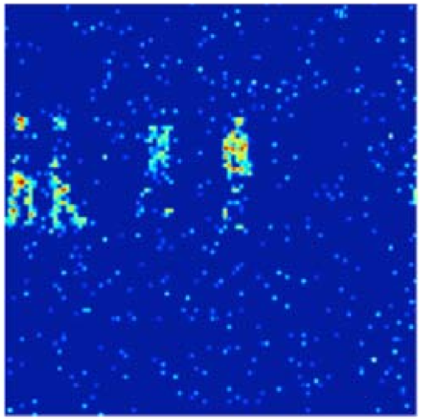




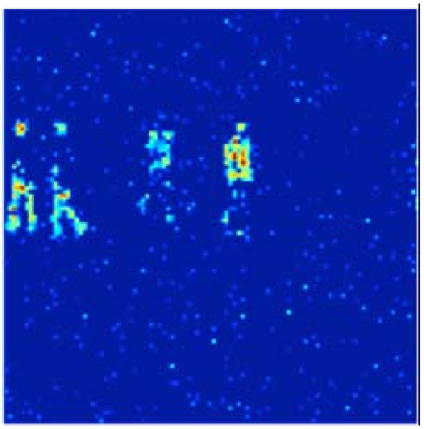
target



Ising-model recovery



CoSaMP recovery



LP (FPC) recovery

Summary

- Why CS works: stable embedding for signals with concise geometric structure

- Sparse signals >> **model-sparse signals**
- Compressible signals >> **model-compressible signals**

- Greedy model-based signal **recovery algorithms**

upshot: provably fewer measurements
more stable recovery

new concept: RIP >> **RAmP**

New Directions

- Diverse data types demand **new models**
  - deterministic models
  - probabilistic/Bayesian/graphical models [Carin et al]
  - manifold models for signal ensembles [Wakin et al]

- New model-based recovery **algorithms**

- Can we **weaken RAmP**?

- Relate to results in coding/info theory literature [Nowak et al, ...]

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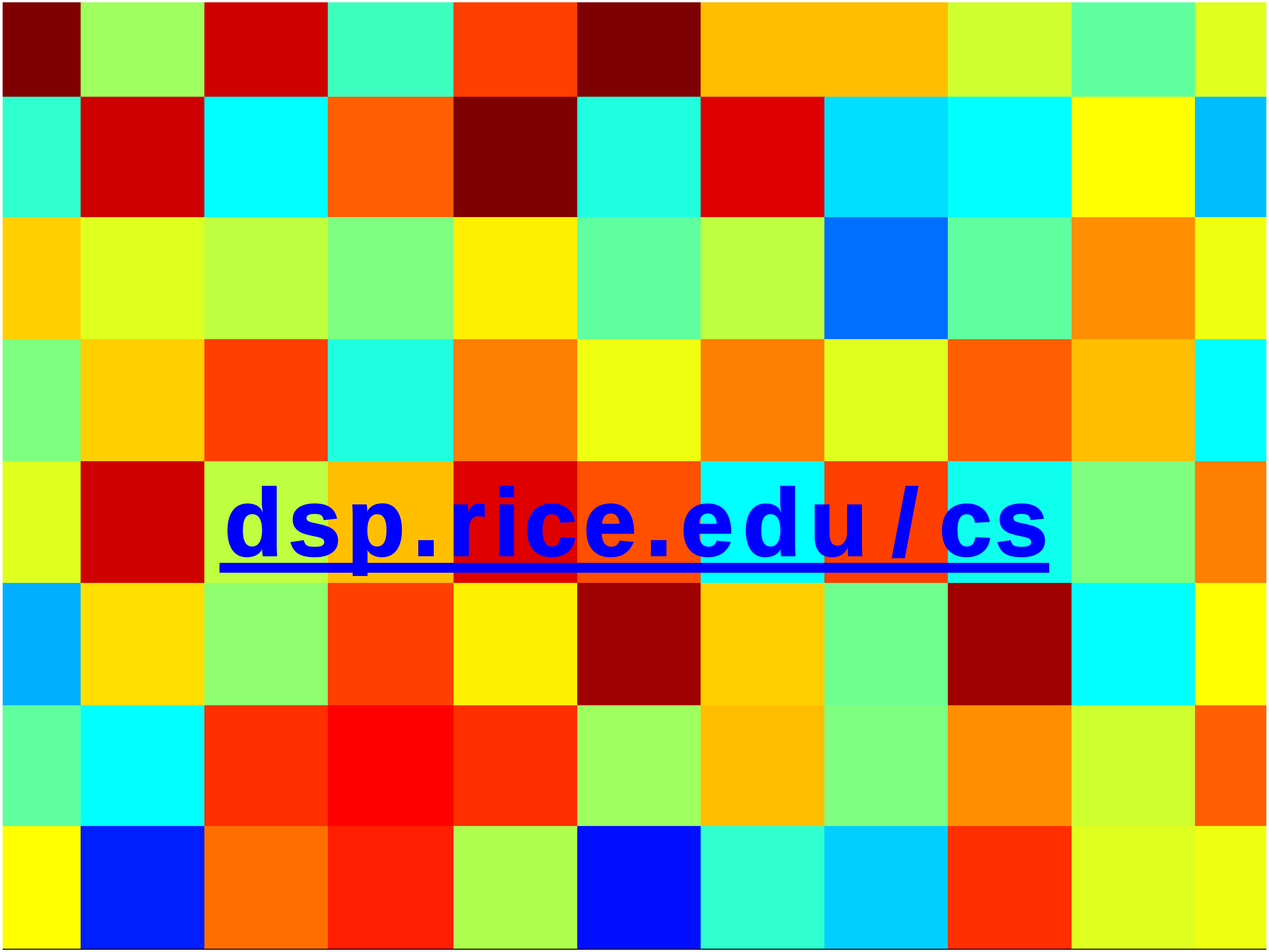
Open Positions

open postdoc positions in sparsity / compressive sensing at Rice University

dsp.rice.edu

richb@rice.edu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