Lévy Measure Decompositions for the Beta and Gamma Processes

(Supplementary material)

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A1. Lévy measure decomposing of beta process

$$
\begin{aligned}
\nu(d\pi, d\omega) &= c(\omega)\pi^{-1}(1-\pi)^{c(\omega)-1}d\pi\mu(d\omega) \\
&= c(\omega)\sum_{k=0}^{\infty}(1-\pi)^{k}(1-\pi)^{c(\omega)-1}d\pi\mu(d\omega) \\
&= \sum_{k=0}^{\infty}\frac{\Gamma(c(\omega)+k+1)}{\Gamma(c(\omega)+k)\Gamma(1)}\pi^{1-1}(1-\pi)^{c(\omega)+k-1}d\pi\frac{c(\omega)}{c(\omega)+k}\mu(d\omega) \\
&= \sum_{k=0}^{\infty}\text{Beta}(1, c(\omega)+k)d\pi\frac{c(\omega)}{c(\omega)+k}\mu(d\omega)
\end{aligned}
$$

A2. Expectation of B_k

For $\forall\mathcal{A}\in\mathcal{F}$, denote $\mathbb{E}_{\text{Poi}}$ and $\mathbb{E}_{\text{Beta}}$ as the expectation computation incurred respectively by μ_k and $\text{Beta}(1, c(\omega)+k)$; and $\{(\omega_{ki}, B_k(\omega_{ki}))\}_i$ a realization of Π_k. Apply restriction theorem and Campbell's theorem [1]:

$$
\begin{aligned}
\mathbb{E}(B_k(\mathcal{A})) &= \mathbb{E}_{\text{Poi}}(\mathbb{E}_{\text{Beta}}(B_k(\mathcal{A}))) \\
&= \mathbb{E}_{\text{Poi}}(\sum_{(\omega_{ki},\, B_k(\omega_{ki}))\,\in\,\Pi_k}\frac{1}{c(\omega_{ki})+k+1}) \\
&= \int_{\mathcal{A}}\frac{1}{c(\omega)+k+1}\mu_k(d\omega)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=0}^{\infty}\mathbb{E}(B_k(\mathcal{A})) &= \int_{\mathcal{A}}\sum_{k=0}^{\infty}\frac{1}{c(\omega)+k+1}\mu_k(d\omega) \\
&= \int_{\mathcal{A}}\sum_{k=0}^{\infty}\frac{c(\omega)}{[c(\omega)+k][c(\omega)+k+1]}\mu(d\omega) \\
&= \int_{\mathcal{A}}\sum_{k=0}^{\infty}c(\omega)[\frac{1}{c(\omega)+k}-\frac{1}{c(\omega)+k+1}]\mu(d\omega) \\
&= \int_{\mathcal{A}}\mu(d\omega) = \mu(\mathcal{A}) = \mathbb{E}(B(\mathcal{A}))
\end{aligned}
$$

A3. Variance of B_k

For the variance, first calculate $\mathbb{E}(B_k^2(\mathcal{A}))$:

$$\begin{aligned}\mathbb{E}(B_k^2(\mathcal{A})) =& \mathbb{E}_{\text{Poi}}(\mathbb{E}_{\text{Beta}}(\sum_{(\omega_{ki},\, B_k(\omega_{ki}))\,\in\,\Pi_k} B_k^2(\omega_{ki}) + \sum_{\substack{(\omega_{ki},\, B_k(\omega_{ki}))\,\in\,\Pi_k \\ (\omega_{ki'},\, B_k(\omega_{ki'}))\,\in\,\Pi_k \\ \omega_{ki}\,\neq\,\omega_{ki'}}} B_k(\omega_{ki})B_k(\omega_{ki'}))) \\ =& \mathbb{E}_{\text{Poi}}[\sum_{\omega_{ki},\, B_k(\omega_{ki}))\,\in\,\Pi_k} \frac{2}{(c(\omega_{ki})+k+1)(c(\omega_{ki})+k+2)} \\ &+ \sum_{\substack{(\omega_{ki},\, B_k(\omega_{ki}))\,\in\,\Pi_k \\ (\omega_{ki'},\, B_k(\omega_{ki'}))\,\in\,\Pi_k \\ \omega_{ki}\,\neq\,\omega_{ki'}}} \frac{1}{(c(\omega_{ki})+k+1)(c(\omega_{ki'})+k+1)}]\end{aligned}$$

The first term $I_1 = \int_{\mathcal{A}} \frac{2}{(c(\omega)+k+1)(c(\omega)+k+2)}\mu_k(d\omega)$ by applying Campbell's theorem; For the second term I_2:

$$\begin{aligned}I_2 =& \sum_{n=0}^{\infty} \frac{\exp(-\int_\Omega \mu_k(d\omega))(\int_\Omega \mu_k(d\omega))^n}{n!} \underbrace{\int_{\mathcal{A}} \cdots \int_{\mathcal{A}}}_{n} \Pi_{i=1}^n [\frac{\mu_k(d\omega_{ki})}{\int_\Omega \mu_k(d\omega)}]\cdot \\ & \sum_{\substack{(\omega_{ki},\, B_k(\omega_{ki}))\,\in\,\Pi_k \\ (\omega_{ki'},\, B_k(\omega_{ki'}))\,\in\,\Pi_k \\ \omega_{ki}\,\neq\,\omega_{ki'}}} \frac{1}{(c(\omega_{ki})+k+1)(c(\omega_{ki'})+k+1)} \\ =& \sum_{n=0}^{\infty} \frac{\exp(-\int_\Omega \mu_k(d\omega))(\int_\Omega \mu_k(d\omega))^n}{n!}(n^2-n)\cdot \\ & \int_{\mathcal{A}}\int_{\mathcal{A}} \frac{\mu_k(d\omega_{ki})\mu_k(d\omega_{ki'})}{(\int_\Omega \mu_k(d\omega))^2} \frac{1}{(c(\omega_{ki})+k+1)(c(\omega_{ki'})+k+1)} \\ =& (\int_{\mathcal{A}} \frac{1}{c(\omega)+k+1}\mu_k(d\omega))^2\end{aligned}$$

Thus: $\text{Var}(B_k(\mathcal{A})) = \mathbb{E}(B_k^2(\mathcal{A})) - (\mathbb{E}(B_k(\mathcal{A})))^2 = \int_{\mathcal{A}} \frac{2}{(c(\omega)+k+1)(c(\omega)+k+2)}\mu_k(d\omega)$

$$\begin{aligned}\sum_{k=0}^{\infty} \text{Var}(B_k(\mathcal{A})) &= \int_{\mathcal{A}} \sum_{k=0}^{\infty} \frac{2c(\omega)}{(c(\omega)+k)(c(\omega)+k+1)(c(\omega)+k+2)}\mu(d\omega) \\ &= \int_{\mathcal{A}} \sum_{k=0}^{\infty} c(\omega)[\frac{1}{c(\omega)+k} - \frac{2}{c(\omega)+k+1} + \frac{1}{c(\omega)+k+2}]\mu(d\omega) \\ &= \int_{\mathcal{A}} \frac{1}{c(\omega)+1}\mu(d\omega) = \text{Var}(B(\mathcal{A}))\end{aligned}$$

A4. Truncation analysis of beta process

$$\begin{aligned}
\text{RHS of (15)} &= 1 - \mathbb{E}[\Pi_{k=K+1}^{\infty}\Pi_{i=1}^{n_k}(1-\pi_{ki})^M] \\
&\overset{(a)}{\leq} 1 - \{\Pi_{k=K+1}^{\infty}\mathbb{E}[\Pi_{i=1}^{n_k}(1-\pi_{ki})]\}^M \\
&= 1 - \{\Pi_{k=K+1}^{\infty}\mathbb{E}[\mathrm{e}^{\sum_{i=1}^{n_k}\log(1-\pi_{ki})}]\}^M \\
&\overset{(b)}{=} 1 - \{\Pi_{k=K+1}^{\infty}[\mathrm{e}^{\int_{\Omega\times(0,1)}\mathrm{e}^{\log(1-\pi)}\nu_k(d\pi,d\omega)}]\}^M \\
&= 1 - \mathrm{e}^{-M\int_\Omega\sum_{k=K+1}^{\infty}\frac{1}{c+K+1}\mu_k(d\omega)} \\
&= 1 - \mathrm{e}^{-M\int_\Omega \mu_{K+1}(d\omega)}
\end{aligned}$$

where (a) is justified by Jensen's inequality and (b) the Campbell's theorem.

By the Euler-Maclaurin formula,

$$\frac{\mathbb{E}(I_K)}{\gamma} = \sum_{k=0}^{K}\frac{1}{1+\frac{k}{c}} \approx c\cdot\log(1+\frac{K}{c}) + \frac{1}{2}(1+\frac{1}{1+\frac{K}{c}}) \overset{K\to\infty}{\approx} c\cdot\log(1+\frac{K}{c})$$

so $K \approx c(\mathrm{e}^{\frac{\mathbb{E}(I_K)}{c\gamma}} - 1)$. Thus the $\mathcal{L}_1$ distance: $\frac{c}{c+K+1} \approx \mathrm{e}^{-\frac{\mathbb{E}(I_K)}{c\gamma}}$. For the stick-breaking construction of [2], $(\frac{c}{c+1})^{K+1} \approx (1+\frac{1}{c})^{-\frac{\mathbb{E}(I_K)}{\gamma}}$. With $c\to\infty$, $(1+\frac{1}{c})^c \uparrow \mathrm{e}$, thus $\mathrm{e}^{-\frac{\mathbb{E}(I)}{c\gamma}} < (1+\frac{1}{c})^{-\frac{\mathbb{E}(I)}{\gamma}}$.

A5. Limit of the Lévy measure of IBP

Since $\nu_{\text{IBP}} = \frac{N}{\gamma}\text{Beta}(c\frac{\gamma}{N}, c)d\pi\mu(d\omega) = \frac{N}{\gamma\Gamma(c\frac{\gamma}{N})}\pi^{c\frac{\gamma}{N}-1}(1-\pi)^{c-1}d\pi\mu(d\omega)$, to prove (22) is equal to prove $\frac{\gamma}{N}\Gamma(c\frac{\gamma}{N}) \overset{N\to\infty}{=} \frac{1}{c}$:

$$\begin{aligned}
\frac{\gamma}{N}\Gamma(c\frac{\gamma}{N}) &= \Delta\int_0^\infty \mathrm{e}^{-t}t^{c\Delta-1}dt, \quad \text{with } \Delta = \frac{\gamma}{N} \\
&= \frac{1}{c}\int_0^\infty \mathrm{e}^{-t}(c\Delta t^{c\Delta-1})dt \\
&= \frac{1}{c}(\mathrm{e}^{-t}t^{c\Delta}|_0^\infty - \int_0^\infty -\mathrm{e}^{-t}t^{c\Delta}dt) \\
&\overset{N\to\infty}{=} \frac{1}{c}
\end{aligned}$$

A6. Lévy measure decomposing of gamma process

For the Lévy measure of the gamma process $\nu(dp,d\omega) = p^{-1}\mathrm{e}^{-\frac{p}{\theta(\omega)}}dp\alpha(d\omega)$, decompose the exponential part into: $\mathrm{e}^{-\frac{p}{\theta(\omega)}} = \mathrm{e}^{\frac{p}{\theta(\omega)}}\mathrm{e}^{-\frac{p}{\theta(\omega)/2}}$, and apply Taylor series expansion on $\mathrm{e}^{\frac{p}{\theta(\omega)}}$:

$$\nu(dp, d\omega) = p^{-1}[\sum_{h=0}^{\infty} \frac{(\frac{p}{\theta})^h}{h!}]\mathrm{e}^{-\frac{p}{\theta/2}} dp\alpha(d\omega)$$
$$= [p^{-1} + \sum_{h=1}^{\infty} \frac{p^{h-1}}{\theta^h h!}]\mathrm{e}^{-\frac{p}{\theta/2}} dp\alpha(d\omega)$$
$$= \sum_{h=1}^{\infty} \mathrm{Gamma}(h, \theta/2) dp \frac{\alpha(d\omega)}{2^h h} + p^{-1}\mathrm{e}^{-\frac{p}{\theta/2}} dp\alpha(d\omega)$$

thus $G = \Gamma_1 + \Gamma\mathrm{P}(\alpha, \theta(\omega)/2)$. Further decompose the exponential part of the gamma process $\Gamma\mathrm{P}(\alpha, \theta(\omega)/2)$ yields $G = \Gamma_1 + \Gamma_2 + \Gamma\mathrm{P}(\alpha, \theta(\omega)/3)$. Keep on this manipulation:

$$G = \sum_{k=1}^{\infty} \Gamma_k, \quad \Gamma_k = \sum_{h=1}^{\infty} \Gamma_{kh}$$
$$\nu_{kh} = \mathrm{Gamma}(h, \frac{\theta}{k+1}) dp \frac{\alpha(d\omega)}{(k+1)^h h}$$

with Γ_{kh} a Lévy process with ν_{kh} its Lévy measure. Here $\mathrm{Gamma}(h, \frac{\theta}{k+1})$ is the PDF of Gamma distribution with shape h and scale $\frac{\theta}{k+1}$.

A7. The expectation of Γ_k and Γ_{kh}

For $\forall \mathcal{A} \in \mathcal{F}$, with Campbell's theorem applied,

$$\mathbb{E}(\Gamma_{kh}(\mathcal{A})) = \int_{\mathcal{A}} \frac{h\theta(\omega)}{k+1} \frac{\alpha(d\omega)}{(k+1)^h h} = \frac{\int_{\mathcal{A}} \theta(\omega)\alpha(d\omega)}{(k+1)^{h+1}}$$
$$\mathbb{E}(\Gamma_k(\mathcal{A})) = \sum_{h=1}^{\infty} \mathbb{E}(\Gamma_{kh}(\mathcal{A})) = \frac{\int_{\mathcal{A}} \theta(\omega)\alpha(d\omega)}{k(k+1)}$$
$$\mathbb{E}(G(\mathcal{A})) = \sum_{k=1}^{\infty} \mathbb{E}(\Gamma_k(\mathcal{A})) = \int_{\mathcal{A}} \theta(\omega)\alpha(d\omega)$$

A8. The variance of Γ_k and Γ_{kh}

The variance of Γ_{kh} can be calculated with the method described in section A3:

$$\mathrm{Var}(\Gamma_{kh}(\mathcal{A})) = \int_{\mathcal{A}} \frac{(h+1)\theta^2(\omega)}{(k+1)^{h+2}} \alpha(d\omega)$$

$$
\begin{aligned}
\mathrm{Var}(\Gamma_k(\mathcal{A})) &= \sum_{h=1}^{\infty} \mathrm{Var}(\Gamma_{kh}(\mathcal{A})) \\
&= \sum_{h=1}^{\infty} \int_{\mathcal{A}} \frac{(h+1)\theta^2(\omega)}{(k+1)^{h+2}} \alpha(d\omega) \\
&= \int_{\mathcal{A}} [\sum_{h=1}^{\infty} -\frac{\theta^2(\omega)}{(k+1)^{(h+1)}}]'_k \alpha(d\omega) \\
&= [\frac{1}{k^2} - \frac{1}{(k+1)^2}] \int_{\mathcal{A}} \theta^2(\omega)\alpha(d\omega)
\end{aligned}
$$

$$
\mathrm{Var}(G(\mathcal{A})) = \sum_{k=1}^{\infty} \mathrm{Var}(\Gamma_k(\mathcal{A})) = \int_{\mathcal{A}} \theta^2(\omega)\alpha(d\omega)
$$

References

[1] J. F. C. Kingman. *Poisson Processes*, volume 3 of *Oxford Studies in Probability*. Oxford University Press, Oxford, 1993.

[2] John William Paisley, Aimee K. Zaas, Christopher W. Woods, Geoffrey S. Ginsburg, and Lawrence Carin. A stick-breaking construction of the beta process. In Johannes Frnkranz and Thorsten Joachims, editors, *ICML*, pages 847–854. Omnipress, 2010.