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# Supplement to Tree-Structured Infinite Sparse Factor Model

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## 1 Proof of Theorem 1

To prove  $\Omega_{\mathbf{b}}$  converges *a.s.* for any  $\mathbf{b} \in \mathcal{T}$  we need to show that matrix  $\Delta_{\mathbf{b}} = \sum_{n \in \mathbf{b}^i} \mathbf{d}_n \mathbf{d}_n^T$  converges *a.s.* for each element  $\sum_{n \in \mathbf{b}^i} d_{rl} d_{ls}$ ,  $1 \leq r, s \leq P$ .

**Lemma 1.** (*Lévy's theorem*) Suppose  $\{X_n\}_{n \geq 1}$  is an independent sequence of random variables,  $\sum_{n=1}^{\infty} X_n$  converges *a.s.* iff  $\sum_{n=1}^{\infty} X_n$  converges *i.p.*

**Lemma 2.** (*Cauchy criterion*)  $\{X_n\}_{n \geq 1}$  converges *i.p.* iff  $X_{n+k} - X_k \rightarrow 0$  *i.p.* as  $n, k \rightarrow \infty$

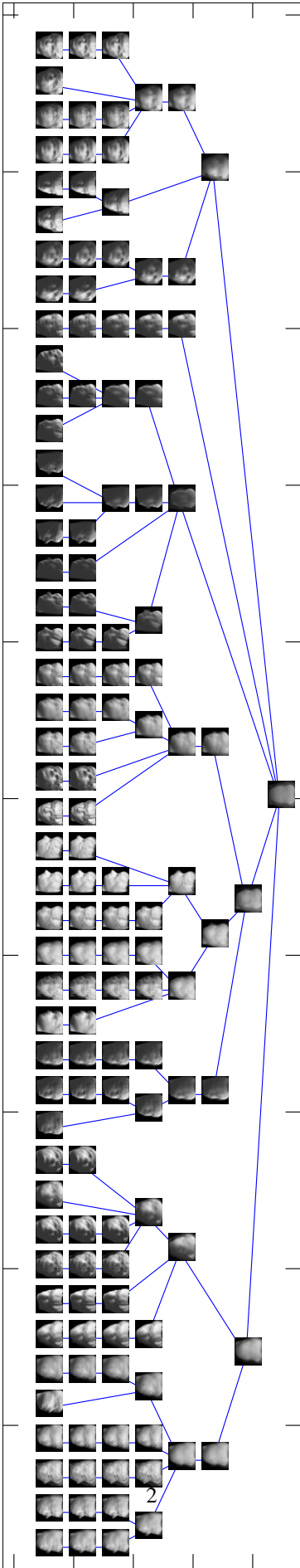
Since each branch  $\mathbf{b}$  is modeled as a factor model, for all loadings on  $\mathbf{b}$  we denote  $m_{k,p} = \sum_{l=k}^{\infty} d_{pl}^2$ . By Cauchy-Schwartz inequality:  $(\sum_{l=k}^{\infty} d_{rl} d_{ls})^2 \leq \sum_{l=k}^{\infty} d_{rl}^2 \sum_{l=k}^{\infty} d_{ls}^2 \leq \max_{1 \leq p \leq P} m_{k,p}^2$ . Combining this result with Lemma 1,2, to prove Theorem 1 it's sufficient to show that  $\lim_{k \rightarrow \infty} p(\max_{1 \leq p \leq P} m_{k,p} < \epsilon) = 1$ :

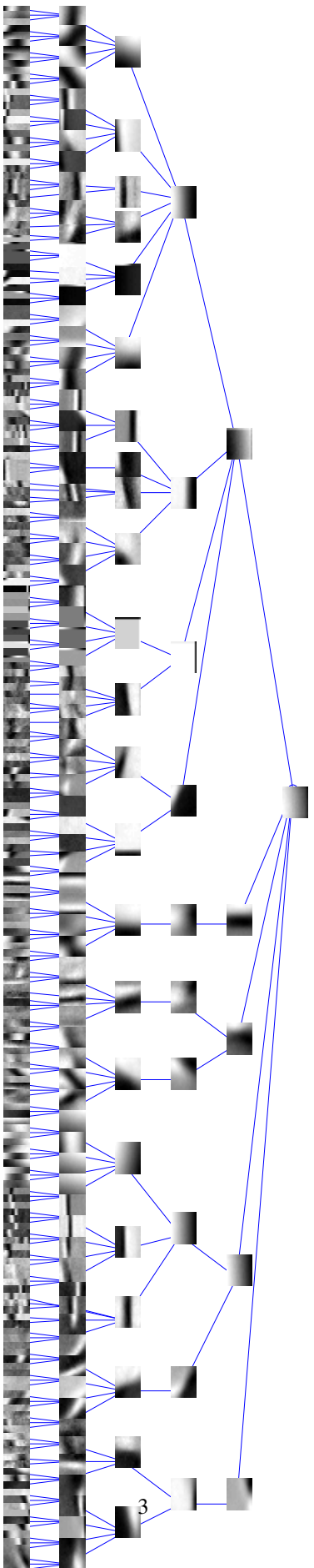
$$\begin{aligned} p(\max_{1 \leq p \leq P} m_{k,p} < \epsilon) &= E\{p(\max_{1 \leq p \leq P} m_{k,p} < \epsilon | \gamma_p)\} = E\{p(m_{k,1} < \epsilon | \gamma_1)^P\} \\ &\geq E\{p(m_{k,1} < \epsilon | \gamma_1)\}^P = (1 - E\{p(m_{k,1} \geq \epsilon | \gamma_1)\})^P \geq \left(1 - E\left(\frac{E(m_{k,1} | \gamma_1)}{\epsilon}\right)\right)^P \end{aligned}$$

where the equality in the first line follows from the fact that  $m_{k,p}$  are conditionally i.i.d given  $\gamma_p$  and the subsequent two inequalities use Jensen's and Chebyshev's inequality respectively. Now based on equation (3) in the paper:  $E(E(m_{k,1} | \gamma_1)) = \sum_{l=k}^{\infty} 3ba^{l-1} = \frac{3b}{1-a} a^{k-1}$ , where  $b = E(\zeta_1^{-1})$ ,  $a = E(\zeta_2^{-1}) < 1$  if  $c_2 > 2$ . At last we arrive at the sufficiency equation and thus Theorem 1 is proved:

$$\lim_{k \rightarrow \infty} p(\max_{1 \leq p \leq P} m_{k,p} < \epsilon) \geq \lim_{k \rightarrow \infty} \left(1 - \frac{3b}{(1-a)\epsilon} a^{k-1}\right)^P = 1$$

## 2 Figure 1 and Figure 4





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