Supplement to Tree-Structured Infinite Sparse Factor Model

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1 Proof of Theorem 1

To prove $\boldsymbol{\Omega_b}$ converges $a.s.$ for any $\boldsymbol{b} \in \mathcal{T}$ we need to show that matrix $\boldsymbol{\Delta_b} = \sum_{n \in \boldsymbol{b}^i} \boldsymbol{d}_n \boldsymbol{d}_n^T$ converges a.s. for each element $\sum_{n \in \boldsymbol{b}^i} d_{rl} d_{ls}, 1 \leq r, s \leq P$.

Lemma 1. *(Lévy's theorem) Suppose $\{X_n\}_{n \geq 1}$ is an independent sequence of random variables, $\sum_{n=1}^{\infty} X_n$ converges a.s. iff $\sum_{n=1}^{\infty} X_n$ converges i.p.*

Lemma 2. *(Cauchy criterion) $\{X_n\}_{n \geq 1}$ converges i.p. iff $X_{n+k} - X_k \to 0$ i.p. as $n, k \to \infty$*

Since each branch $\boldsymbol{b}$ is modeled as a factor model, for all loadings on $\boldsymbol{b}$ we denote $m_{k,p} = \sum_{l=k}^{\infty} d_{pl}^2$. By Cauchy-Schwartz inequality: $\left(\sum_{l=k}^{\infty} d_{rl} d_{ls}\right)^2 \leq \sum_{l=k}^{\infty} d_{rl}^2 \sum_{l=k}^{\infty} d_{ls}^2 \leq \max_{1 \leq p \leq P} m_{k,p}^2$. Combining this result with Lemma 1,2, to prove Theorem 1 it's sufficient to show that $\lim_{k \to \infty} p(\max_{1 \leq p \leq P} m_{k,p} < \epsilon) = 1$:

$$p(\max_{1 \leq p \leq P} m_{k,p} < \epsilon) = E\{p(\max_{1 \leq p \leq P} m_{k,p} < \epsilon | \boldsymbol{\gamma}_p)\} = E\{p(m_{k,1} < \epsilon | \boldsymbol{\gamma}_1)^P\}$$
$$\geq E\{p(m_{k,1} < \epsilon | \boldsymbol{\gamma}_p)\}^P = (1 - E\{p(m_{k,1} \geq \epsilon | \boldsymbol{\gamma}_1)\})^P \geq \left(1 - E\left(\frac{E(m_{k,1}|\boldsymbol{\gamma}_1)}{\epsilon}\right)\right)^P$$

where the equality in the first line follows from the fact that $m_{k,p}$ are conditionally i.i.d given $\boldsymbol{\gamma}_p$ and the subsequent two inequalities use Jensen's and Chebyshev's inequality respectively. Now based on equation (3) in the paper: $E(E(m_{k,1}|\boldsymbol{\gamma}_1)) = \sum_{l=k}^{\infty} 3ba^{l-1} = \frac{3b}{1-a} a^{k-1}$, where $b = E(\zeta_1^{-1}), a = E(\zeta_2^{-1}) < 1$ if $c_2 > 2$. At last we arrive at the sufficiency equation and thus Theorem 1 is proved:

$$\lim_{k \to \infty} p(\max_{1 \leq p \leq P} m_{k,p} < \epsilon) \geq \lim_{k \to \infty} \left(1 - \frac{3b}{(1-a)\epsilon} a^{k-1}\right)^P = 1$$

2 Figure 1 and Figure 4