Interferometric Array Imaging

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Outline

- Numerical simulations to assess algorithms for detecting and imaging underground structures with passive and active acoustic arrays.

- Review and critique of time reversal imaging (Kirchhoff migration) when there is clutter.

- Review and critique of incoherent interferometry (matched field imaging).

- Introduction of coherent interferometry (CINT) based on the decoherence frequency and length.

- A resolution theory for CINT.

- Critique of CINT and introduction of adaptive CINT or ACINT.

- The ACINT research program.
Array imaging: the data

Active array data: $P(x_s, x_r, t)$ for $(x_s, x_r, t)$ a set of source-receiver locations in $R^2 \times R^2$ and time in $R_+$. A five-dimensional parametrization of the data. Passive: $P(x_r, t)$, a three-dimensional dataset.

Different data acquisition geometries: Synthetic aperture imaging (zero-offset, large linear apertures, broadband), Ultrasonic imaging arrays (many sources and receivers, broadband signals).

Look carefully at resolution and noise issues.
Computational domain $100\lambda_0 \times 100\lambda_0$ with central wavelength $\lambda_0 = 3m$ (at central frequency $f_0 = 1KH\bar{z}$ and with $c_0 = 3km/sec$), surrounded by a perfectly matched layer (pink).

The array has 185 receiving elements $\lambda_0/2$ apart, for an aperture of $92\lambda_0$. 

The three sources are ultrawideband (more than 100% bandwidth) at a distance of $90\lambda_0$ from the array. The near ones are $6\lambda_0$ apart and the far one is $3\lambda_0$ behind the near ones.

The random fluctuations of the propagation speed, on the right, have 3% STD and a Gaussian correlation (monoscale) function. The correlation length is equal to the central wavelength $\lambda_0 = 3m$.

We use a FINITE ELEMENT TIME DOMAIN CODE (2D or 3D), resolving all scales with 30 – 40 points per wavelength. It takes about two hours on a workstation to produce a set of (2D) synthetic data.

We have well established 3D elastic and electromagnetic FETD codes. The codes are now being ported on a computer cluster that was obtained recently.
Passive array: traces

Down: Homogeneous, 1%, 3%STD.
Active array or echo-mode imaging: traces

Down: homogeneous, \( s = 1\% \) and 2\% STD.
Time reversal imaging or Kirchhoff migration

The imaging functional for KM at a search point $y^S$ is:

$$I^{KM-PASS}(y^S) = \sum_r P(x_r, \tau(x_r, y^S)) = \sum_r \frac{1}{2\pi} \int d\omega \hat{P}(x_r, \omega) e^{-i\omega \tau(x_r, y^S)}$$

$$I^{KM-ACT}(y^S) = \sum_r P(x_r, x_r, 2\tau(x_r, y^S)) = \sum_r \frac{1}{2\pi} \int d\omega \hat{P}(x_r, x_r, \omega) e^{-i\omega 2\tau(x_r, y^S)}$$

where $\tau(x, y) = |x - y|/c_0$ is the travel time in a homogeneous background with propagation speed $c_0$.

Well-known result for deterministic (smooth) media: Up to using a weight factor in the array sum, as the aperture increases to all of $R^2$ and the bandwidth to all of $R$, $I^{KM}(y^S) \sim \delta(y^S - y)$ (single target).

In KM-ACT only zero offset is used. Non-zero offset data used typically for velocity estimation and/or (here) for clutter reduction.
Critique of time reversal imaging

How does time reversal imaging or Kirchhoff Migration (KM) (or Synthetic Aperture Imaging) work?

It does not work well in clutter.

It is statistically unstable in clutter.

The reason is that KM tries to cancel the random phase of the signals arriving at the array with a deterministic phase using travel times.

The true Green’s function for the random medium is not known and so cannot be used for imaging, which would result is huge resolution enhancement as in physical TR.

KM is inherently unstable statistically, which is a very serious flaw.
Time reversal imaging results (passive array)

Down: Homogeneous, 1%, 2%, 3%STD. Across: different realizations.
Incoherent Interferometry

To avoid the random phase problems in Kirchhoff migration imaging we mimic physical time reversal by computing cross-correlations of data traces, the interferograms, and summing

\[ I^{INT}(y^S) = \sum_{x_r,x_r'} P(x_r, \cdot) *_t P(x_r', -\cdot)|_{\tau(x_r,y^S) - \tau(x_r',y^S)} \]

The interferograms are given by

\[ P(x_r, \cdot) *_t P(x_r', -\cdot)(t) = \int_{-\infty}^{\infty} P(x_r, s)P(x_r', s - t)ds \]

The interferograms are self-averaging. We are doing Differential Kirchhoff Migration on the lag of the interferograms, or time reversal of correlations.
In the frequency domain we have

\[ I^{INT}(y^S) = \int d\omega \left| \sum_{x_r} \hat{P}(x_r, \omega) e^{-i\omega \tau(x_r, y^S)} \right|^2 \]

This is almost Matched Field Imaging

\[ I^{MF}(y^S) = \int d\omega \left| \sum_{x_r} \hat{P}(x_r, \omega) \tilde{G}_0(x_r, y^S, \omega) \right|^2, \quad \tilde{G}_0(x, y, \omega) = \frac{e^{i\omega \tau(x, y)}}{4\pi|x - y|} \]

that is widely used in sonar and elsewhere in more general situations (waveguides, enclsures, etc) with a suitable \( \tilde{G}_0 \).
Decoherence distance $\Delta_d$ and decoherence frequency $\Omega_d$

Time traces at $x_r$ and $x_{r'}$ decorrelate rapidly in a random medium as the distance between $x_r$ and $x_{r'}$ increases. That is, the trace cross-correlation

$$P(x_r, \cdot) \ast_{t} P(x_{r'}, -\cdot)(t)$$

go to zero as $|x_r - x_{r'}|$ increases.

The decoherence distance $\Delta_d$ is related to the effective aperture in time reversal, which is essentially independent of any physical aperture. It can be ESTIMATED from the array data directly and $0 \leq \Delta_d \leq a$ with $a$ the aperture of the array.

The phases of $\hat{P}(x_r, \omega_1)$ and $\hat{P}(x_r, \omega_1)$ decorrelate when $|\omega_1 - \omega_2|$ increases. The decoherence frequency $\Omega_d$ is related to the delay spread (reverberation) in the traces due to the random inhomogeneities. It can be ESTIMATED from the array data directly and $0 \leq \Omega_d \leq B$ with $B$ the bandwidth of the probing pulse.
Critique of incoherent interferometry

- Incoherent interferometric imaging is effective only if the decoherence frequency $\Omega_d$ is very small, essentially zero relative to the bandwidth $B$. Frequency coherence is not used!

- Incoherent interferometric imaging gets depth resolution only by triangulation, which needs large arrays. Sometimes an arrival time analysis can give depth resolution but there is no robust way to do this.

- Incoherent interferometry gets direction of arrival (angular) resolution from which the decoherence length $\Delta_d$ can also be ESTIMATED. Broadband is essential here (Inverse Problems, vol 19, (2003), pp. 5139-5164).
Coherent interferometric imaging

To combine good random phase cancellation, which KM does not have but INT and MF have, with exploitation of residual coherence effects we introduce the Coherent Interferometric functional:

\[
I^{CINT}(y^S) = \int \int |\omega_1 - \omega_2| \leq \Omega_d \, d\omega_1 d\omega_2 \sum \sum |x_r - x'_r| \leq \Delta_d \, 
\hat{P}(x_r, \omega_1) \hat{P}(x'_r, \omega_2) e^{-i(\omega_1 \tau(x_r, y^S) - \omega_2 \tau(x'_r, y^S))}
\]

Using midpoint (sum) and offset (difference) variables: \( x_r = \bar{x} - \tilde{x}/2, \ x'_r = \bar{x} + \tilde{x}/2 \) and \( \omega_1 = \bar{\omega} - \tilde{\omega}/2, \ \omega_1 = \bar{\omega} + \tilde{\omega}/2 \) we rewrite it as

\[
I^{CINT}(y^S) = \int d\tilde{\omega} \sum \bar{x} \int |\tilde{\omega}| \leq \Omega_d \, \sum |\bar{x}| \leq \Delta_d \, 
\hat{P}(\bar{x} - \frac{\tilde{x}}{2}, \bar{\omega} - \frac{\tilde{\omega}}{2}) \hat{P}(\bar{x} + \frac{\tilde{x}}{2}, \bar{\omega} + \frac{\tilde{\omega}}{2}) e^{i\tilde{\omega}[\tau(\bar{x} + \frac{\tilde{x}}{2}, y^S) - \tau(\bar{x} - \frac{\tilde{x}}{2}, y^S)]} e^{\frac{i}{2} \tau(\bar{x} - \frac{\tilde{x}}{2}, y^S) + \tau(\bar{x} + \frac{\tilde{x}}{2}, y^S)}
\]
Using the space-wavenumber, time-frequency Wigner function of the data

\[ W_D(\bar{x}, p; \bar{t}, \bar{\omega}) = \int d\bar{\omega} \sum_{|\bar{\omega}| \leq \Omega_d} \hat{P}(\bar{x} - \frac{\bar{x}}{2}, \bar{\omega} - \frac{\bar{\omega}}{2}) \hat{P}(\bar{x} + \frac{\bar{x}}{2}, \bar{\omega} + \frac{\bar{\omega}}{2}) e^{i(p \bar{x} + \bar{t} \bar{\omega})} \]

and simplifying, the **coherent interferometric functional** is

\[ I^{CINT}(y^S) = \int d\bar{\omega} \sum_{\bar{x}} W_D(\bar{x}, \bar{\omega} \nabla \bar{x} \tau(\bar{x}, y^S); \tau(\bar{x}, y^S), \bar{\omega}) \]

This is the imaging functional that we use in the results shown.

Note the dependence of \( I^{CINT} \) on the decoherence frequency \( \Omega_d \) and the decoherence distance \( \Delta_d \), which are known in advance.
Coherent interferometric imaging results I

Down: 0%, 1%, 3% STD. Across: $\Omega_d = 0, 10, 20\%$. All: $\Delta_d = 20\%$
Down: 0%, 1%, 3% STD. Across: $\Omega_d = 0, 10, 20\%$. All: $\Delta_d = 40\%$. 

Coherent interferometric imaging results II
• The first column shows the results of incoherent interferometry where frequency coherence is ignored. As a result range resolution is not good, as expected.

• Even a small amount of frequency coherence makes a significant difference in resolution.

• The decoherence frequency $\Omega_d$ is not known and it is adjusted heuristically in the previous two slides, as is the decoherence distance $\Delta_d$. 
No $\bar{\omega}$ averaging. Three realizations at 3% STD. Top: $\Omega_d = 10\% \Delta_d = 20\%$. Bottom: $\Omega_d = 10\% \Delta_d = 40\%$. No statistical stability when bandwidth is not used fully.
CINT imaging results: role of array size

Sum of three $\bar{x}$’s. Three realizations at 3% STD. Top: $\Omega_d = 10\%$ $\Delta_d = 20\%$. Bottom: $\Omega_d = 10\%$ $\Delta_d = 40\%$. Some statistical stability when array is not used fully.
A resolution theory can be developed based on several assumptions about the random medium and the propagation regime.

Such assumptions are NOT used in the numerical simulations.

- With the paraxial approximation, the white noise limit, and a high frequency expansion we reduce all theoretical calculations to the use of one relatively simple formula obtained from the random Schrödinger equation: a second order moment formula.

- One other regime where analytical results can be obtained: Layered media. In no other regime do we have, or expect, analytical results.
Resolution theory II

- The main results are: (a) the resolution in range is $c_0/\Omega_d$ with $c_0$ the homogeneous propagation speed, and (b) the angular or direction of arrival resolution is $(k_0\Delta_d)^{-1}$ with $k_0$ the central wavenumber. We assume here that the SNR at the array is high (essentially infinite).

- Note that in a deterministic medium $\Omega_d = B$, the bandwidth, and $\Delta_d = a$, the array aperture. Loss of resolution is directly tied to space and frequency decoherence.

- How can we estimate accurately $\Omega_d$ and $\Delta_d$? Use multiresolution analysis (the Local Cosine Transform) on the data.
Summary of the resolution theory for passive array imaging

Starting from first principles and in a carefully controlled scaling limit we find that

\[ I^{\text{CINT}}(y^S, \Omega_d, \Delta_d) \approx \int_{\text{Aprt}} d\bar{x} \int_{\text{Bndw}} d\bar{\omega} |\hat{f}_B(\bar{\omega} - \omega_0)|^2 e^{-(\nabla_{\bar{z}}|\bar{x} - y^S| - \nabla_{\bar{z}}|\bar{x} - y|)^2/2\sigma_{\text{DoA}}^2} e^{-(|\bar{x} - y^S| - |\bar{x} - y|)^2/2\sigma_R^2} \]

Here: \( \sigma_{\text{DoA}} = \frac{c_0}{\omega_0 \Delta_d} \), \( \sigma_R = \frac{c_0}{\Omega_d} \) are the direction of arrival and range resolution, respectively, \( \hat{f}_B \) is the Fourier transform of the base-band pulse (depending on the bandwidth \( B \)), and \( \omega_0 \) is the central frequency. We have assumed here that there is only one source at location \( y \).

The decoherence length and frequency, \( \Delta_d \) and \( \Omega_d \), are related to the statistical properties of the random medium in an explicit manner (not shown here).
Comments on the theoretical CINT formula

• Although the theoretical formula is derived for a particular class of random media, it has a general and rather intuitive form as the integral of a Gaussian point spread function.

• The theoretical results are extended to distributed sources and to the active-array case with distributed reflectivity.

• The CINT theory is consistent with our previous incoherent interferometric results, which correspond to the case $\Omega_d = 0$.

• The theory can be used to further process the image by deblurring, as explained at a later slide.
Adaptive Selection of $\Delta_d$ and $\Omega_d$

For CINT to be effective we need to be able to determine the decoherence length and frequency adaptively. We do this as follows.

The imaging functional is

$$I^{CINT}(y^S) = \int \int d\omega_1 d\omega_2 \sum \sum \hat{F}(x_r, \omega_1, y^S) \hat{F}(x_r', \omega_2, y^S)$$

with $\hat{F}(x_r, \omega, y^S) = \hat{P}(x_r, \omega) e^{-i\omega \tau(x_r, y^S)}$

Given the box space-frequency box size, we calculate it by

$$I^{CINT}(h_x, h_\omega, y^S) \approx \int_{Ap't} dx \int_{Bndw} d\omega_1 \int \int_{N(\omega_1, x)} dx' d\omega_2 \hat{F}(x_r, \omega_1, y^S) \bar{\hat{F}}(x_r', \omega_2, y^S)$$
The adaptive CINT algorithm

To select \( h_x \) and \( h_\omega \) we minimize an appropriate norm of the normalized image

For \( h_x, h_\omega \in [0, a] \times [0, B] \), the normalized image is

\[
S(h_x, h_\omega, y^S) = \frac{\sqrt{I(h_x, h_\omega, y^S)}}{\max_{y^S \in D} \sqrt{I(h_x, h_\omega, y^S)}},
\]

where \( I = I^{CINT} \) and the norm used is

\[
|||S|||_D = ||S||_{L^1(D)} + ||\nabla S||_{L^1(D)}
\]

\[
= \int_D dy^S \left( |S(h_x, h_\omega, y^S)| + |\nabla S(h_x, h_\omega, y^S)| \right)
\]

The decoherence frequency and length are directly related to the optimal window size (\( \Omega_d = 2h_\omega \), \( X_d = 2h_d \)). The total variation norm performs well because it preserves features of the image, that is, it does not smooth excessively.
Adaptive coherent interferometric results (I)

Adaptive coherent interferometric results (II)

Adaptive coherent interferometric results (III)

Active array (one illumination). Top: Coherent interferometry. Bottom: Kirchhoff Migration. Across: different realizations SDT 3\% Mono-scale correlation function
Active array (one illumination). Top: Coherent interferometry. Bottom: Kirchhoff Migration. Across: different realizations SDT 3% Multi-scale correlation function
Comments on the adaptive CINT results

- In passive array imaging, when adaptive CINT is compared carefully (in a fair way) to KM the results appear to be not so different.

- Adaptive CINT performs much better than KM in single illumination, active array imaging.

- With adaptive CINT the image does appear to be a blurred version of the original, as the theory predicts.

- In the next slides we show that with three illuminations in active array imaging, KM improves considerably and is comparable to adaptive CINT.
We can use multiple illuminations to improve the signal to noise ratio due to the clutter. This is called post-stack migration (coherent summing after back-propagation) and consists in replacing

\[ \hat{F}(x_s, x_r, \omega, y^S) = \hat{P}(x_s, x_r, \omega) e^{-i\omega(\tau(x_s, y^S) + \tau(y^S, x_r))} \]

by

\[ \sum_{x_s} \hat{F}(x_s, x_r, \omega, y^S) \]

where \( x_s \) are the illumination points.

We then follow the same algorithm as before.
Adaptive coherent interferometric results (V)

Active array. Top: Coherent interferometry. Bottom: Kirchhoff Migration. Across: different realizations SDT 3% Mono-scale correlation function
• Complete the non-adaptive CINT theory (active arrays, distributed reflectivity, full array illumination) along with the necessary numerical simulations.

• Explore further and put in some definitive form the adaptive CINT theory, which we have begun recently. The adaptive CINT algorithm improves the quality of the images dramatically. When coupled with the theoretical results, adaptive CINT imaging can be followed by a deblurring step that will give even better images.

• Use the adaptive CINT algorithm to the Gatech GPR data. The non-adaptive CINT algorithm is expected to be quite inadequate with these data.

• Continue the exploration of multi-resolution analysis methods to obtain more general versions of adaptive CINT, using the local cosine transform in particular.

• Extend the CINT theory and algorithms to near-field imaging.
Concluding remarks

- Coherent interferometry, which is the backpropagation of local cross-correlations of traces, is a very effective way to deal with cluttered environments in array imaging and remote sensing.

- Adaptive estimation of the space-frequency decoherence in the array data addresses very well the difficult issue of implementing coherent interferometry and makes the CINT algorithm much more robust.