Imaging for distributed sensors networks

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Position of the problem

We considere a *complex medium*:

![Diagram of a complex medium with sensors labeled 1, 2, ..., q and a sensor firing](image)

**We have some knowledge of this medium** in the following sense:

For a given set of $N$ distributed sensors, the **Response Matrix of the medium** is gathered:

\[
\text{Sensor } p \text{ is firing} \\
\text{Echos are measured at all sensors } q = 1, \ldots, N \quad \rightarrow \quad P_{pq}^0(t)
\]
Suppose now that there is an *object* in the medium:

The **Response Matrix with object** $P_{pq}^d(t)$ is gathered with *the same* set of $N$ distributed sensors

The **difference Response Matrix** $P_{pq}(t) = P_{pq}^d(t) - P_{pq}^0(t)$ contains the echos coming from the objects only.
There is a very important literature in array imaging:

- Migration in geophysics: Claerbout (1976), Bleistein et al. (2001),…
- Interferometry: Zebker (1986), Borcea et al. (2003),…

Relatively few references in distributed sensor imaging:

- Analysis of Lamb wave propagation and triangulation: Lemistre & Balagas (2001), Giurgiutiu et al. (2000)
- Beamforming: Chang et al. (2003)
1. Numerical Setup

2. Kirchhoff Migration

3. Time reversal imaging

4. Separation of the defects by Singular Value Decomposition
Wave equation in 2 dimension in free space. Computational domain $50\lambda \times 50\lambda$, with central wavelength $\lambda = 1\text{cm}$ the reference wave speed $c_0 = 3500 \text{m.s}^{-1}$

The medium contains 25 scatterers with Dirichlet Boundary conditions

12 sensors (4 rows of 3) located at $x_p$, \ p = 1, \ldots N

We want to image 2 pointlike defects with Dirichlet Boundary conditions
About the numerical resolution

We use a Finite Element Time Domain code to solve the 2D wave equation. The space resolution is 32 points per wavelength. Perfectly matched layered (PML) are used to simulate the propagation in free space. It takes about 1:30 hour on a workstation to produce a synthetic Response Matrix.

The probing pulse is ultrawideband: second derivative of a gaussian with central frequency $\nu = 350\text{kHz}$ and $\approx 130\%$ bandwidth:

$$f(t) = (2(\sqrt{2\pi\nu t} - t_0)^2 - 1) \exp(- (\sqrt{2\pi\nu t} - t_0)^2)$$

The probing pulse in the time domain (left) and its Fourier Transform (right)
Sensor # 6 is firing. Normalized traces measured at all sensors.

X-axis: time, Y-axis: number of sensor
Outline

1. Numerical Setup

2. Kirchhoff Migration

3. Time reversal imaging

4. Separation of the defects by Singular Value Decomposition
Kirchhoff Migration

The only knowledge on the background that is used is the reference velocity $c_0$.

With traces measured when sensor # $p$ is firing, compute for any search point $y^S$:

$$I_{p}^{KM}(y^S) = \sum_{q=1}^{N} P_{pq}(\tau_p(y^S) + \tau_q(y^S))$$

where $\tau_q(y^S) = |x_q - y^S|/c_0$ is the travel time from sensor # $q$ to $y^S$.

**Important result:** for a single target at $y^*$ in a homogeneous medium, a large array and large bandwidth, one has $I_{p}^{KM}(y^S) \approx \delta(y^S - y^*)$. Is there a similar result for distributed sensors?

**With arrays:** Range resolution is controled by the bandwidth and cross range resolution is limited by the aperture of the array.

**With sensors:** resolution limited by the bandwidth of the probing pulse and uniformity of distribution of sensors around the target.
Kirchhoff Migration Images

Images with illumination #2, #6 and #10 (second row of sensors)

Highly dependent on the illumination (large spot, only 1 target, ghosts,…)

Kirchhoff Migration is unstable with data that have lot of delay spread and is thus unreliable

Increasing the number of sensors improves the result
1. Numerical Setup

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4. Separation of the defects by Singular Value Decomposition
For imaging, we back propagate numerically the data in an idealized medium.

We assume here that the actual background is perfectly known, to see what best can be achieved with TR imaging.

For each illumination \( \#p = 1, \ldots N \), the back propagated field \( u_p(y, t) \) is solution of:

\[
\begin{cases}
\frac{\partial^2 u_p}{\partial t^2} - c_0^2 \Delta u_p = \sum_{q=1}^{N} \delta(x_q, y) P_{pq}(T - t) & \text{in } \mathbb{R}^2 \setminus \Omega, \\
u_p(y, t) = 0, & \text{on } \delta \Omega,
\end{cases}
\]

where \( \Omega \) is the set of all objects in the background. \( T \) is the recording duration.
Time reversal with echo mode data: forward propagation

(a) sensor $p$ sends a pulse at time $t = 0$

(b) target $y^*$ starts emitting later on at unknown time $t = t^*$ as a secondary source

(c) The traces are recorded at all sensors
Time reversal with echo mode data: Backward propagation

(d) All sensors reemit the time reversed traces

(e) The back propagated field focuses first on the target \( y^* \) at unknown time \( t = t^* \)

(f) In a second time, the back propagated field focuses on the illuminating sensor \( p \) at time \( t = 0 \)
Time reversal imaging with echo mode data

\( u_p(y, t) \) focuses on the defects because they act as secondary sources, so we can make an image by taking a snapshot of the field \( u_p(y, t) \) at the refocusing time.

But we do not know at what time the refocusing occurs, because we do not know where the target is. So the refocusing time will be characterized as the minimum of a sparsity norm of the field:

- Shannon entropy
- Total Variation norm (TV norm)
Definition of Shannon entropy

Let \( u_{ij} \) denote the space discretized field \( u_p \) at time \( t \), containing \( N_p \) pixels. Given a number of gray levels \( N_c \), the histogram of frequencies of colors of the image \( u_{ij} \) is:

\[
h_c = \sum_{i,j} \delta_c(u_{ij})
\]

where \( \delta_c \) is the counting function of the gray level \( c \).

Example: the histogram of

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
\text{Index of gray level} & 0 & 50 & 100 & 150 & 200 & 250 & 300 \\
\hline
\text{Number of occurences} & 20 & 15 & 10 & 20 & 8 & 3 & 2
\end{array}
\]

The entropy is defined by:

\[
S(u_{ij}) = - \sum_c \frac{h_c}{N_p} \log_2 \frac{h_c}{N_p}
\]

The Shannon entropy is a measure of the information needed to encode a discrete image. Its penalizes images that have a high speckle level.
Definition of the TV norm

It is the norm of the space of functions of bounded variation. Roughly speaking this space contains functions that are integrable as well as their gradient.

For a discrete image $u_{ij}$, the TV norm is defined by:

$$TV(u_{ij}) = h^2 \sum (|\tilde{u}_{ij}| + |\nabla_h \tilde{u}_{ij}|),$$

where $\tilde{u}_{ij} = u_{ij} / \max(|u_{ij}|)$ and $\nabla_h$ is a low order finite difference approximation of the gradient with space step $h$.

The TV norm is widely used in image processing (deblurring,…) for its good ability to detect or preserve sharp contours of objects.

It penalizes images with lot of oscillations
Time reversal images using optimal entropy stopping

Entropy versus time – illumination #2, #6 and #10 (second row of sensors)

Backpropagated field at minimum of entropy
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A model for the Response Matrix

Assume there are $M$ isotropic pointlike targets located at $y_k^*$. If we neglect multiple reflections, a model of the Response Matrix in the frequency domain is given by:

$$\hat{P}_{pq}^{mod}(\omega) = \hat{f}(\omega) \sum_{k=1}^{M} \tau_k(\omega) \hat{G}(x_p, y_k^*, \omega) \hat{G}(y_k^*, x_q, \omega),$$

$\hat{G}(x_p, y_k^*, \omega)$ is the Transfer Function (or Green Function) from $x_p$ to $y_k^*$ of the actual background and $\tau_k(\omega)$ is the scattering amplitude of the $k^{th}$ target.

Let $\hat{g}_k(\omega)$ denote the illuminating vector coming from the target $y_k^*$:

$$\hat{g}_k(\omega) = \left[ \hat{G}(y_k^*, x_1, \omega), \hat{G}(y_k^*, x_2, \omega), \ldots, \hat{G}(y_k^*, x_N, \omega) \right]^T$$

Then rewrite $\hat{P}$ as:

$$\hat{P}_{pq}^{mod}(\omega) = \hat{f}(\omega) \sum_{k=1}^{M} \tau_k(\omega) \hat{g}_k(\omega) \hat{g}_k^T(\omega),$$
Singular Value Decomposition of the Response Matrix

\[ \hat{P}(\omega) = \sum_{k=1}^{N} \sigma_k^2(\omega) \hat{U}_k(\omega) \hat{V}_k^H(\omega), \]

\( (\hat{U}_k(\omega)) \) is an orthonormal basis of eigenvectors of the hermitian matrix \( \hat{P}(\omega) \hat{P}^H(\omega) \) associated to the real positive eigenvalues \( \sigma_k^2(\omega) \).

By comparison with the model, the number of leading singular values gives the number of targets:

\[ \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_M \gg \sigma_{M+1} \approx \ldots \approx \sigma_N \approx 0. \]

The first 3 singular values of the response matrix \( \hat{P}(\omega) \) versus frequency

Separation of the targets using the SVD

We say that the targets are well resolved if:

$$\hat{g}_k^H(\omega)\hat{g}_l(\omega) \approx 0, \ \forall k \neq l.$$  

then the leading singular vectors are proportional to the illuminating vectors:

$$\hat{U}_k(\omega) \approx e^{i\phi_k(\omega)} \frac{\hat{g}_k(\omega)}{|\hat{g}_k(\omega)|}.$$  

So in principle, the SVD transforms an echo-mode problem into an active source problem.

But the singular vector $U_k(t)$ looks incoherent in the time domain, because of the arbitrary phase $\phi_k(\omega)$. 

Separation of the targets using the SVD

To get rid of the arbitrary phase, we construct $N$ different versions of the eigenvectors by projecting each column of the Response Matrix into it:

$$\hat{U}^{(p)}_k(\omega) = [\hat{U}^H_k(\omega) \hat{P}^{(p)}(\omega)] \hat{U}_k(\omega), \quad p = 1, \ldots N,$$

One has:

$$\hat{U}^{(p)}_k(\omega) \approx \hat{f}(\omega) \hat{\tau}_k(\omega) \hat{G}(x_p, y^*_k, \omega) g_k(\omega)$$

So $\hat{U}^{(p)}_k(\omega)$ corresponds to the column obtained when the $k^{th}$ defect is alone.

The time domain singular vector is then given by:

$$U_k^{(p)}(t) = \frac{1}{2\pi} \int e^{-i\omega t} \sigma(\omega) \hat{U}_k^{(p)}(\omega) d\omega$$

It is possible to apply any of the above imaging algorithms to the Response Matrix projected on each singular vectors.
Optimal entropy stoping - Projection on the first singular vector

Entropy versus time – illumination #2, #6 and #10 (second row of sensors)

Backpropagated field at minimum of entropy
Optimal entropy stoping - Projection on the second singular vector

Entropy versus time – illumination #2, #6 and #10 (second row of sensors)

Backpropagated field at minimum of entropy
Cumulated images

We can improve the Signal to Noise Ratio by cumulating the images.

KM

TR

Raw traces  1st sing. vec.  2nd sing. vec.
Conclusion

Kirchhoff Migration is not reliable in a complex background.

**Echo-Mode Time reversal imaging** with optimal compensation using Shannon entropy or TV norm proves to work very well:

- It is **stable**
- Gives a **reliable image of each defect**

**Work in progress:**

- Investigate how other algorithms (Migration, MUSIC,…) perform when (part of ) the actual background is known (broadband *vs.* narrowband),

- Use a coherent interferometry algorithm — he traces have a lot of delay spread, which is a case when interferometry is expected to perform well

- Address the question of optimal illumination

- Address the question of optimal allocation of sensors