A Reference Solution for Thermal Mine Signature Modeling

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ABSTRACT
Predicting the thermal signature of a buried land mine requires modeling the complicated inhomogeneous environment and the structurally complex mine. It is useful, both in checking such models and in making rough calculations of expected signatures, to have an accurate, easily computed solution for a relatively simple geometry. In this paper, a reference solution is presented for the integral equation that governs the temperature distribution. Our solution procedure uses the method of weighted residuals. The problem comprises a homogeneous cylindrical body (the mine model) buried in an infinite homogeneous half space (the soil model) with a planar interface. Using periodic boundary conditions in time at the planar interface, the temperature distribution in the lower-half space is expanded in a Fourier series. A volume integral equation for the Fourier series coefficients is obtained via Green's second identity. The Green's function for the Fourier coefficients is derived and reduced to a computationally efficient form. The integral equation is reduced to a matrix equation, which is then solved for the unknown temperature distribution. The integral equation solution is compared with a finite element model.

Keywords: Thermal infrared imagery, land mines, heat transfer, integral equations, thermal model, numerical simulation

1. INTRODUCTION
Research in infrared (IR) detection of land mines has been ongoing for several decades, and those sensors (especially multi-spectral instruments) are considered promising. Since IR sensors provide information independent of the electrical properties of the target, they complement metal detectors and ground-penetrating radar in a sensor-fused system.

Although many experimental and empirical studies have been performed, the physics that define IR signatures are poorly understood. These signatures are dependent on several factors, including solar insolation, cloud cover, vegetation, surface irregularities, and past meteorological conditions. In order to improve detection performance and to better utilize IR sensors, there is a critical need to understand the effects of these environmental factors.

Recently, combined thermal and radiometric models of IR mine signatures have been proposed, which offer insight into the underlying physics. Sendur and Baertlein used a one-dimensional analytical thermal model to predict the surface temperature distribution over a mine. The land mine is assumed to be buried under a layered-earth heated by natural diurnal sources. The analytical formulation is based on a Fourier analysis of the periodic phenomenon. Closed form expressions were derived for the temperature distribution on the soil surface and at depth. That simple analysis confirmed a number of results that are seen in experimental studies, including the presence of two thermal “cross-over” times, a lag between the solar illumination and the soil temperature, the attenuation of the signature strength with depth, and a marked difference in the signatures of thermally conducting and insulating buried objects. Pregowski and Swiderski developed a two-dimensional model to predict the thermal IR signatures of buried land mines. Those numerical simulations were based on a finite-difference time-domain (FDTD) formulation and they included water transport in the soil medium. Pregowski and Swiderski concluded that a single IR image taken at a single spectral range may not be sufficient for IR detection because of clutter. In a more recent study Sendur and Baertlein developed a three-dimensional thermal and radiometric model to predict IR mine signatures. The thermal model was based on the finite element method (FEM), which is capable of modeling realistically shaped mines and inhomogeneous soil. The radiometric model accounted for both the spectral and spatial character of the surface, and the camera geometry and field of view (FOV) are included.

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Although full three-dimensional solutions to the problem have now appeared, the accuracy and efficiency of such model must be addressed. For this reason, an accurate and easily computed solution for a relatively simple geometry is desired for checking the more detailed models and for making rough calculations. In this paper, we present a reference solution based on an integral-equation formulation. The problem geometry comprises a homogeneous cylindrical body (the mine model) buried in an infinite homogeneous half-space (the soil model) with a planar interface. The solution of the volume integral equation for the Fourier series coefficients is obtained using the method of weighted residuals, a technique better known to researchers in electromagnetics as the “Method of Moments” (MoM).

This work is organized as follows: Sect. 2 provides a description of the three-dimensional heat flow equation, including the convective and radiative boundary conditions at the soil-air interface. In addition, the Fourier series representation of the temperature distribution with appropriate boundary conditions is presented in that section. In Sect. 3 integral representations for the temperature distribution in soil and mine are obtained. The Green’s function for this problem is derived in Sect. 4 and subsequently simplified and converted into a computationally efficient form. The numerical solution procedure for the Fourier series coefficients of the temperature distribution is formulated in Sect. 5. Example results are presented in Sect. 6, and concluding remarks appear in Sect. 7.

2. PHYSICAL DESCRIPTION OF THE PROCESS

The temperature distribution in the soil and mine is described by the three-dimensional heat flow equation which, for piecewise constant media can be written

\[ \nabla^2 T(r, t) - \frac{1}{\kappa(r)} \frac{\partial T(r, t)}{\partial t} = 0, \quad i = s, m \]

in which \( T(r, t) \) [K] is the temperature and \( \kappa \) is the thermal diffusivity, defined as the ratio of the thermal conductivity to the volumetric heat capacity. In this result the subscripts \( s \) and \( m \) refer to the soil and mine, respectively. To complete the thermal problem description, boundary condition at the soil-air interface must be specified. At that boundary convective and radiative heat transfer mechanisms are present. Numerous models have been developed in the literature to represent the heat transfer at an soil-air interface. Those models have been extensively studied in one-dimensional thermal analyses of homogeneous soil over a diurnal cycle,\(^{3,4}\) and they are used here. The net heat flux into the ground, \( \mathcal{F}_{\text{net}} \), can be written as

\[ \mathcal{F}_{\text{net}}(t) = \kappa \frac{\partial T(r, t)}{\partial z} \bigg|_{z=0} = \mathcal{F}_{\text{sun}}(t) + \mathcal{F}_{\text{sky}}(t) - \mathcal{F}_{\text{sh}}(t) - \mathcal{F}_{\text{ge}}(t), \]

where \( \mathcal{F}_{\text{sun}} \) is the incident solar radiation reduced by cloud extinction, atmospheric absorption, soil albedo and the cosine of the zenith angle; \( \mathcal{F}_{\text{sky}} \) is the sky brightness with a correction for cloud cover; \( \mathcal{F}_{\text{sh}} \) is the sensible heat transfer from land to atmosphere due to convection; and \( \mathcal{F}_{\text{ge}} \) is the gray body emission from the soil surface.\(^*\) Expressions for these terms have appear in prior works,\(^{3,4}\) and here we summarize those findings. For the solar flux we write

\[ \mathcal{F}_{\text{sun}}(t) = S_0 C_{\text{sun}} \mathcal{H}(t) \]

in which \( S_0 = 1353 \) [W/m\(^2\)] is the solar constant, \( C_{\text{sun}} \) is a factor that accounts for the reduction in absorbed energy due cloud cover and soil reflectivity, and \( \mathcal{H}(t) \) is the local insolation function, assumed to be periodic over a diurnal cycle. The long-wavelength radiation from the atmosphere, \( \mathcal{F}_{\text{sky}} \), is given as

\[ \mathcal{F}_{\text{sky}}(t) = \sigma T_{\text{sky}}^4(t), \]

where \( \sigma = 5.67 \times 10^{-8} \) [W m\(^{-2}\) K\(^{-4}\)] is the Stephan-Boltzmann constant and \( T_{\text{sky}} \) [K] is an effective sky radiance temperature. The heat loss due to ground radiation is given by Stefan’s Law

\[ \mathcal{F}_{\text{ge}}(t) = \varepsilon \sigma T^4(t, z = 0), \]

where \( \varepsilon \) [unitless] is the mean emissivity of the surface and \( T(t, z = 0) \) is the soil temperature at the soil-air interface. The sensible heat transfer between the surface and atmosphere is approximated by

\[ \mathcal{F}_{\text{sh}}(t) = h(t)(T_{\text{air}}(t) - T(t, z = 0)). \]

\(^*\)In formulating this expression we have ignored evaporation of soil moisture and changes in the state in the medium.
where \( h(t) \) is a convection coefficient which depends strongly on the local wind speed. Empirical models for the air and sky temperatures are

\[
\begin{align*}
T_{\text{air}}(t) &= T_{0,\text{air}} - T_{\text{d}} \cos(2\pi(t - 2)/24) \\
T_{\text{sky}}(t) &= T_{\text{air}}(t)C_{\text{sky}}
\end{align*}
\]

(7)

(8)

where \( t \) is the local time as above, and \( T_{0,\text{air}} \) and \( T_{\text{d}} \) are estimated from local meteorological data available from the National Weather Service, and \( C_{\text{sky}} \) is a constant\(^7\) that depends on the local meteorological conditions.

Using Eqs. (3) and (4)-(6) in Eq. (2) results in the boundary condition at the soil-air boundary. This condition is a nonlinear function of the surface temperature distribution, but over the limited temperature range of interest, the result can be linearized using the technique described by Watson\(^4\) to obtain

\[
\frac{\partial T(r, t)}{\partial z} \approx T(r, t) \left( \frac{1}{k_s} (h(t) + 4\epsilon_0 T_{\text{sky}}^3(t)) - \frac{1}{k_s} (F_{\text{sun}}(t) + h(t)T_{\text{air}}(t) + 4\epsilon_0 T_{\text{sky}}^4(t)) \right).
\]

(9)

The boundary condition at the soil-air interface given by Eq. (2) is time-varying. If the convection coefficient, \( h(t) \), is assumed to be time invariant over the diurnal cycle and approximated by its mean value \( \overline{h} \), then the boundary condition becomes a periodic function at the diurnal rate. This approximation suggests that the temperature distribution, \( T(r, t) \), can be written as

\[
T(r, t) = \sum_{n=-\infty}^{\infty} T_n(r)e^{i\omega nt}
\]

(10)

where \( n \) denotes the \( n \)th component of the Fourier series expansion, \( \omega = 2\pi/(24 \times 60 \times 60) \) is the angular frequency, and \( T_n(r) \) is the \( n \)th (complex) coefficient of the Fourier series expansion. Since \( T(r, t) \) is a real quantity, the coefficients are conjugate symmetric, i.e., \( T_n(r) = T_{-n}^*(r) \) where \( * \) denotes the complex conjugate operator. Substituting Eq. (10) in Eq. (1), the Fourier coefficients of the three-dimensional heat-flux equation can be written as

\[
\nabla^2 T_n(r) - \frac{i\omega n}{k_s(r)} T_n(r) = 0, \quad i = s, m
\]

(11)

Solar heating is the dominant heat source in this problem, and this fact permits some additional approximations. Replacing \( T_{\text{air}}(t) \) with its mean value \( \overline{T_{\text{air}}} = T_{0,\text{air}} \), and simplifying \( T_{\text{sky}} \) in a similar manner yields

\[
\frac{\partial T(r, t)}{\partial z} \approx T(r, t) \left( \frac{\overline{h}}{k_s} + 4\epsilon_0 T_{\text{sky}}^3 \right) - \frac{1}{k_s} \left( F_{\text{sun}}(t) + \overline{h}T_{0,\text{air}} + 4\epsilon_0 T_{\text{sky}}^4 \right)
\]

(12)

By using the Fourier series expansion of the solar insolation function

\[
F_{\text{sun}}(t) = \sum_{n=-\infty}^{\infty} F_n e^{i\omega nt}
\]

(13)

the boundary condition for the \( n \)th Fourier coefficient can be written as

\[
\frac{\partial T_n(r)}{\partial z} = \alpha T_n(r) + \beta_n
\]

(14)

where

\[
\alpha = \frac{\overline{h}}{k_s} + 4\epsilon_0 T_{\text{sky}}^3
\]

(15)

\[
\beta_n = -F_n - \overline{h}T_{0,\text{air}} - 4\epsilon_0 T_{\text{sky}}^4
\]

(16)
Figure 1. A cylindrical mine with radius $\rho_0$ and thickness $t$, buried at depth $h$ under a smooth soil surface.

3. INTEGRAL REPRESENTATION FOR THE TEMPERATURE DISTRIBUTION

An integral representation of the temperature distribution for both soil and mine can be obtained using Green’s second identity with an appropriate Green’s function. In this section such an integral representation is obtained for a cylindrical mine with radius $\rho_0$ and thickness $t$, buried at depth $h$ under a smooth soil surface as shown in Fig. 1. Homogeneous soil with thermal diffusivity $\kappa_s(r) = \kappa_s$ and thermal conductivity $\kappa_s(r) = \kappa_s$ is assumed. The thermal properties of the mine need not be uniform in the following formulation and the position $(\mathbf{r})$ dependence on the mine’s thermal properties will be retained in this formulation.

For a cylindrical mine, the problem has rotational symmetry. Therefore, Eq. (11) can be written as

$$\begin{align*}
\nabla_{\rho z}^2 T_n^s(r) - k_n^2 T_n^s(r) &= 0 \quad ; \quad \mathbf{r} \in \text{soil} \\
\nabla_{\rho z}^2 T_m^m(r) - k^2_m(r)T_m^m(r) &= 0 \quad ; \quad \mathbf{r} \in \text{mine}
\end{align*}$$

(17)

where

$$\begin{align*}
\nabla_{\rho z}^2 &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \frac{\partial}{\partial \rho}) + \frac{\partial^2}{\partial z^2} \\
k_n^2 &= \frac{i \omega n}{\kappa_s} \\
k^2_m(r) &= \frac{i \omega n}{\kappa_m(r)}
\end{align*}$$

(18-20)

In Eq. (17) $T_m^m(r)$ and $T_n^s(r)$ represent the Fourier coefficients in the mine and soil regions, respectively.

We employ a standard method$^{10}$ of solving the differential operator equation defined in Eq. (11). Forming the inner product of that equation and the Green’s function $G$ yields

$$\langle \nabla_{\rho z}^2 T_n, G(\rho, z; \rho', z') \rangle = \langle \nabla_{\rho z}^2 G(\rho, z; \rho', z'), T_n(\rho', z') \rangle + \mathcal{R}(\rho, z)$$

(21)

where $\langle \cdot \rangle$ is the inner product, which consists in integration over the entire lower half space. The Green’s function satisfies the equation

$$\nabla_{\rho z}^2 G(\rho, z; \rho', z') = -\frac{\delta(\rho - \rho')}{\rho} \delta(z - z')$$

(22)

with boundary conditions defined subsequently. The conjunct$^{10} \mathcal{R}$ is an integral over the problem boundaries and is also defined below. Using the above property of $G$ and the differential equation satisfied by $T_n$ we immediately obtain

$$T_n(\rho, z) = \int_{z_1}^{z_2} dz' \int_0^{\rho_0} d\rho' \rho' c(\rho', z') T_m^m(\rho', z') G(\rho, z; \rho', z') + \mathcal{R}$$

(23)

In this result $z_1 = h$ and $z_2 = h + t$, we have exchanged primed and unprimed coordinates, and we have defined

$$c(\mathbf{r}) = k^2_m(\mathbf{r}) - k^2_n = \frac{i \omega n}{\kappa_s} \frac{\kappa_s - \kappa_m(\mathbf{r})}{\kappa_s \kappa_m(\mathbf{r})}$$

(24)
Integration by parts is used to evaluate $\mathcal{R}$ from its definition in Eq. (21). In doing so, we exploit the thermal boundary conditions, namely that the temperature is continuous

$$T_n^m (\mathbf{r} = \mathbf{r}_{ms}) = T_n^s (\mathbf{r} = \mathbf{r}_{ms})$$

and the thermal fluxes are continuous

$$\mathcal{K}_m \nabla T_n^m (\mathbf{r}) \bigg|_{\mathbf{r} = \mathbf{r}_{ms}} = \mathcal{K}_s \nabla T_n^s (\mathbf{r}) \bigg|_{\mathbf{r} = \mathbf{r}_{ms}}$$

where $\mathbf{r}_{ms}$ denotes the mine-soil boundary. The following result is obtained

$$\mathcal{R}(\rho, z) = -\int_0^\infty dp' \int_{\rho}^{\infty} dp'' \mathcal{Z}'(\rho, z; \rho', 0) + \left(1 - \frac{\mathcal{K}_m}{\mathcal{K}_s}\right) \left(\rho_0 \int_{z_1}^{z_2} dz' \mathcal{F}'(\rho, z; \rho_0, z') \right)$$

$$+ \int_0^{\rho_0} dp' \int_{\rho_0}^{\infty} dp'' \left(S^m(\rho, z; \rho', z_2) - S^m(\rho, z; \rho', z_1)\right)$$

$$\bigg|_{z'=z'}$$

In Eqs. (28)-(30) the subscript $i$ can be $s$ and $m$ which refer to the soil and mine, respectively. By selecting the boundary condition for the Green’s function at the soil-air interface as

$$\frac{\partial G(\rho, z; \rho', z')}{\partial z} = \alpha G(\rho, z; \rho', z')$$

the integral representation for the temperature distribution $T_n(\rho, z)$ in the lower half space can be obtained as

$$T_n(\rho, z) = -\int_{z_1}^{z_2} dz' \int_0^{\infty} dp' \int_{\rho}^{\infty} dp'' \mathcal{Z}'(\rho, z; \rho', 0)T_n^m(\rho, z; \rho', z')G(\rho, z; \rho', z')$$

$$- \beta_n \int_0^{\rho_0} dp' \rho' G(\rho, z; \rho', z' = 0) + \left(1 - \frac{\mathcal{K}_m}{\mathcal{K}_s}\right) \left(\rho_0 \int_{z_1}^{z_2} dz' \mathcal{F}'(\rho, z; \rho_0, z') \right)$$

$$+ \int_0^{\rho_0} dp' \int_{\rho_0}^{\infty} dp'' \left(S^m(\rho, z; \rho', z_2) - S^m(\rho, z; \rho', z_1)\right)$$

Equation (32) is an integral relation from which one can determine the temperature distribution anywhere in the lower half space by integrating the temperature distribution over the mine. If the point $(\rho, z)$ is within the mine, this relation becomes a Fredholm integral equation of the second kind for the unknown mine temperature distribution, i.e., the unknown function appears both inside and outside of the integral. In the absence of the mine, i.e., for a homogeneous half space, the first and last terms in Eq. (32) vanish, and the temperature distribution can be found directly from

$$T_n(\rho, z) = -\beta_n \int_0^{\infty} dp' \rho' G(\rho, z; \rho', z' = 0)$$

(33)
4. GREEN’S FUNCTION

The Green’s function $G$ is the solution of the heat transfer equation for an internal point source of heat. It must satisfy Eq. (22) with the boundary condition on the soil-air interface given by Eq. (31). In addition to this boundary condition, the Green’s function must have a finite value for $\rho = 0$ and must vanish at $\rho \to \infty$ and $z \to \infty$. In Sect. 4.1 $G$ is derived, and in Sect. 4.2 $G$ is transformed into a form more appropriate for numerical evaluation.

4.1. Derivation of Green’s Function

The Green’s function is easily derived in the spectral domain. Taking the Hankel transform\textsuperscript{10} of both sides of the Eq. (22) we obtain a one-dimensional Green’s function problem

$$
\left( \frac{d^2}{dz^2} - \left( k^2_n + k^2_\rho \right) \right) G(k_\rho, z; z') = - J_0(k_\rho \rho') \delta(z - z')
$$

(34)

in which $k_\rho$ is the spectral variable and $J_0$ is the Bessel function of the first kind. The solution of this equation is readily obtained using standard methods\textsuperscript{10} and after inverting the Hankel transform we obtain

$$
G(\rho, z; \rho', z') = \int_0^\infty dk_\rho J_0(k_\rho \rho') J_0(k_\rho \rho) \frac{k_\rho}{2 \sqrt{k^2_n + k^2_\rho}} \exp \left( - \sqrt{k^2_n + k^2_\rho} z_+ \right)
\times \left[ \exp \left( \sqrt{k^2_n + k^2_\rho} z_- \right) + r_0(k_n, k_\rho, \alpha) \exp \left( - \sqrt{k^2_n + k^2_\rho} z_- \right) \right]
$$

(35)

where $z_+$ ($z_-$) is the greater (lesser) of $z$ and $z'$, and $r_0$ is a thermal “reflection coefficient” given by

$$
r_0(k_n, k_\rho, \alpha) = \frac{\sqrt{k^2_n + k^2_\rho} - \alpha}{\sqrt{k^2_n + k^2_\rho} + \alpha}
$$

(36)

4.2. Simplification of Green’s Function

Efficient, accurate calculation of $G$ is crucial to a numerical solution of the integral equation, but in general, numerical evaluation of the Green’s function is computationally challenging. The major difficulties in the computation of this integral is that (1) for large $\rho$ and $\rho'$ values, the functions $J_0(k_\rho \rho)$ and $J_0(k_\rho \rho')$ oscillate rapidly, leading to slow convergence; and (2) for $|z - z'| \ll 1$ or $(z + z') \ll 1$ the exponential terms in the integrals decay slowly, again producing slow convergence.

The integral given by Eq. (35) is similar to the Sommerfeld integrals\textsuperscript{11}, which have been studied extensively in the physics and electromagnetics literature. Straightforward manipulations ameliorate some of its problems. The reflection coefficient $r_0(k_n, k_\rho, \alpha)$ can be expressed as

$$
r_0(k_n, k_\rho, \alpha) = 1 - \frac{2\alpha}{\sqrt{k^2_n + k^2_\rho} + \alpha}
$$

(37)

and, hence, the Green’s function can be represented as

$$
G(\rho, z; \rho', z') = G_1(\rho, z; \rho', z') + G_2(\rho, z; \rho', z') + G_3(\rho, z; \rho', z')
$$

(38)

where

$$
G_1(\rho, z; \rho', z') = \int_0^{2\pi} d\phi' \frac{\exp(ik_\rho R_1)}{4\pi R_1} \frac{k_\rho}{\sqrt{k^2_n + k^2_\rho}}
$$

(39)

$$
G_2(\rho, z; \rho', z') = \int_0^{2\pi} d\phi' \frac{\exp(ik_\rho R_2)}{4\pi R_2} \frac{k_\rho}{\sqrt{k^2_n + k^2_\rho}}
$$

(40)

$$
G_3(\rho, z; \rho', z') = -\alpha \int_0^\infty dk_\rho \frac{k_\rho J_0(k_\rho \rho') J_0(k_\rho \rho)}{\sqrt{k^2_n + k^2_\rho} \sqrt{k^2_n + k^2_\rho}} \exp \left( - \sqrt{k^2_n + k^2_\rho} (z + z') \right)
$$

(41)
in which we have defined

\[ R_1 = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi' + (z - z')^2} \]  \hspace{1cm} (42)

\[ R_2 = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi' + (z + z')^2} \]  \hspace{1cm} (43)

and we have used the following identities\textsuperscript{12,10}

\[ J_0(k\rho')J_0(k\rho) = \frac{1}{2\pi} \int_0^{2\pi} J_0(k\rho\rho_\ell) d\phi \]  \hspace{1cm} (44)

\[ \exp(-k\sqrt{\rho^2 + \rho'^2}) \]  \hspace{1cm} (45)

where

\[ \rho_\ell = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi} \]  \hspace{1cm} (46)

The form of \( G_3 \) can be simplified by using a device suggested by Kuo and Mei.\textsuperscript{13} Omitting the details in the interest of brevity, we obtain

\[ G_3(\rho, z; \rho', z') = -\frac{\alpha}{2\pi} \int_0^{2\pi} d\phi \exp(-\alpha(z + z')) \int_{0}^{\infty} \frac{\exp(ik_\ell R_3)}{R_3} \exp(-\alpha\zeta)d\zeta \]  \hspace{1cm} (47)

where

\[ R_3 = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos\phi + \zeta^2} \]  \hspace{1cm} (48)

The last integral in Eq. (47) has infinite integration limits, but the integrand is a rapidly decaying function for typical values of \( \alpha \), and it is well approximated by an integral over a finite domain. When the source and observation points approach one another, we have \( R_1 \to 0 \), and the Green’s function has a well-known integrable singularity. The treatment of this case is is discussed in Sect. 5.2.

5. NUMERICAL SOLUTION OF THE INTEGRAL EQUATION

In this section we describe a numerical solution procedure for the integral equation in Eq. (32) the yield the Fourier coefficients \( T_n^m(\mathbf{r}) \). Our approach is to solve for the temperature \( T_n^m \) within the mine. Given the mine’s temperature distribution, we then evaluate Eq. (32) to find the Fourier coefficients \( T_n(\rho, z = 0) \) for the temperature distribution at the surface. The time history of the temperature is then found by evaluating the Fourier series.

We employ the method of weighted residuals (MWR) in this work. The electromagnetics community has developed an extensive body of knowledge on MWR solution of integral equations in the temporal frequency domain under the guise of the MoM. The MoM discretizes the integral equation into a matrix equation, the solution of which is obtained using standard methods. Following the pioneering works of Richmond\textsuperscript{14} and Harrington,\textsuperscript{15} an extensive literature has developed on this procedure, and summary references are also available\textsuperscript{16-18}

5.1. Solution Procedure

The solution of the integral equation using the MoM begins with a discretization of the unknown temperature distribution over the buried mine with expansion functions. It is important to select expansion functions appropriate for the problem at hand. The integral representation of the temperature distribution given in Eq. (32) requires both the temperature and its derivatives with respect to \( \rho \) and \( z \). In this work we employ subsectional expansion function \( \Lambda(\rho, z) \) that are linear in \( \rho \) and \( z \) with the result

\[ T_n^m(\rho, z) = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{m,n} \Lambda_{mn}(\rho, z) \]  \hspace{1cm} (49)

where \( M \) and \( N \) are the number of divisions in the \( z \) and \( \rho \) directions. This representation yields a continuous, piecewise-linear representation of \( T_n^m \) and a piecewise constant form for its derivatives.
Figure 2. An example discretization of the buried mine with 4 subdivisions in the \( \hat{\rho} \)-direction and 3 subdivisions in the \( \hat{z} \)-direction.

A sample discretization of the buried mine with \( N = 4 \) subdivisions in the \( \hat{\rho} \)-direction and \( M = 3 \) subdivisions in the \( \hat{z} \)-direction is illustrated in Fig. 2. The thermal properties over each subsectional basis function are assumed constant and taken into account in the constant \( c_{mn} \) which can be defined as

\[
c_{mn} = c \left( \left( \rho_n + \rho_{n-1} \right) / 2, \left( z_m + z_{m-1} \right) / 2 \right)
\]  

(50)

where \( c \) was previously defined with Eq. (24) and the indexed values of \( \rho \) and \( z \) are the subsection boundaries. By substituting the discretized temperature distribution given by Eq. (49) into the integral equation Eq. (32), we obtain

\[
\beta_n \int_0^\infty dp' dp' G_n p, z; \rho', z' = 0 = - \sum_{m=1}^M \sum_{n=1}^N A_{mn} \left[ \Lambda_{nm} p, z, \rho', z' \Lambda_{nm} p', z' \right] + \left( 1 - \frac{K_m}{K_s} \right) \rho_0 \sum_{m=1}^M A_{mN} \int dz' G_n p, z; \rho' = \rho_0, z', \rho' = \frac{d}{dz'} \Lambda_{nm} p', z' + \sum_{n=1}^N A_{Nn} \int dp' dp' G_n p, z; \rho', z' = z_2 \frac{d}{dz'} \Lambda_{nm} p', z' + \sum_{n=1}^N A_{1n} \int dp' dp' G_n p, z; \rho', z' = z_1 \frac{d}{dz'} \Lambda_{nm} p', z'
\]  

(51)

To complete the formulation, we multiply both sides of Eq. (51) by testing functions \( w_p p, z \) and integrate over the mine. The result is a linear system of equations

\[
\mathbf{V} = \mathbf{Z} \mathbf{A}
\]  

(52)

for the constants \( A_{mn} \) which we represent by the matrix \( \mathbf{A} \). Solving Eq. (52) using any convenient scheme yields the unknown coefficient vector \( \mathbf{A} \) which can be used to compute the temperature distribution over the buried mine via Eq. (49). If desired, the temperature distribution everywhere in the lower half-space can be computed using Eq. (32). The details of computing the matrix elements \( Z_{pq} \) are presented in Sects. 5.2.

Various testing functions have been used in the literature. Galerkin methods are known to be optimum in the least-square sense, but they are also computationally expensive in terms of matrix-fill time. To reduce the computational cost, we have employed a point-matching technique, for which the testing functions are delta functions at appropriately chosen points in each subsection.

5.2. Computation of Z Matrix

In this section evaluation of the matrix elements \( Z_{pq} \) is discussed. To make the notation more concise, we use the index \( p \) to refer to a particular point in \( \rho, z \), which we denote \( \rho_{pq}, z_{pq} \) with a similar convention for \( q \). In general, \( Z_{pq} \) involves integrals of the form

\[
\int_{z_{pq}^{-1}}^{z_{pq}} dz' \int_{\rho_{pq}^{-1}}^{\rho_{pq}} dp' G_n p, z; \rho', z' f(p', z')
\]  

(53)
where \( f(\rho, z) \) is polynomial function of the arguments. Consider first the off-diagonal matrix elements \( Z_{pq} \) with \( p \neq q \). For this case, the testing point \( (\rho_n, z_m) \) is not in the integration domain and the Green's function is a smooth function throughout the integration domain. Therefore, \( Z_{pq} \) can be directly evaluated by numerical methods.

For the diagonal matrix elements \( Z_{pp} \), however, the testing point \( (\rho_n, z_m) \) lies in the integration domain. This results in \( R_1 \rightarrow 0 \) which causes a singularity as \( R_1^{-1} \). Consequently, an integrand regularization technique (e.g., a singularity extraction technique) is necessary in numerical evaluation of the integrals. The singularity extraction technique is widely used in MoM solutions in electromagnetics for both surface and volume formulations.\(^{19-22}\) Singularity extraction techniques of the Green's functions for bodies of revolution have also been studied in the electromagnetics literature.\(^{23,24}\)

In brief, calculation of the diagonal elements requires that the singular parts of the integrand are evaluated separately by analytical methods. A careful examination suggests that the singular part of the Green's function \( G(\rho, z; \rho', z') \) is due to function \( G_1(\rho, z; \rho', z') \), which leads us to write

\[
G_1(\rho, z; \rho', z') = \int_0^{2\pi} d\phi' \frac{\exp(i k_n R_1) - 1}{4 \pi R_1} f(\rho', z') + \int_0^{2\pi} d\phi' \frac{1}{4 \pi R_1} f(\rho', z')
\]

The first integral is evaluated numerically. The second term is integrated analytically in a small region about the singularity, and integration over the remainder of the domain is done numerically. Details may be found in the literature.

The numerical procedure described in Sect. 5.1 yields the Fourier coefficients of the temperature distribution within the mine. The temperature distribution over the soil surface \( z = 0 \) is found by using Eqs. (32) and (51) to obtain the Fourier coefficients at the surface. Some numerical problems arise in the surface integrals, since \( z + z' = 0 \), but an approximate, accurate, and efficient form for the Green’s function is

\[
G(\rho, z = 0; \rho', z' = 0) \approx \frac{1}{2\pi} \int_0^{2\pi} d\phi \frac{\exp(i k_n \rho_1)}{\rho_1} - \alpha K_0(-i k_n \rho_1) I_0(-i k_n \rho_1)
\]

\[
- \frac{\alpha^2}{2\pi} \int_0^{2\pi} d\phi \frac{\exp(i k_n \rho_1)}{ik_n} - \frac{\alpha^2}{4\pi} \int_0^{2\pi} d\phi' \frac{\rho_1 K_{-1}(-i k_n \rho_1)}{ik_n}
\]

The first term in Eq. (51) can be integrated numerically using the Green’s function expression in Eq. (47). For the second term in Eq. (51), the Green’s function representation given by Eq. (55) is used.

6. RESULTS

In this section we present example results that illustrate the accuracy of this solution procedure. Although the ultimate test of any model is its ability to predict measurements, uncertainty in environmental parameters makes it difficult to obtain a meaningful comparison. As a substitute, we compare our results with both analytical formulations and with FEM results. Other than the heat transfer equation and the assumed parameter values, these methods have little in common with one another, which makes the comparison informative.

We begin with a calculation of the temperature over a homogeneous half space performed using three different techniques, namely, an analytical model, an integral-equation based model, and an FEM based model. The problem is modeled using a half-space extending to infinity with constant thermal properties. The analytical solution is based the result of Watson,\(^4\) which uses a Fourier series solution of the heat transfer equation.\(^{25}\) In Fig. 3, the predicted soil surface temperature is given as a function of time. The integral equation solution and the FEM solution are in agreement with the analytical results, which lends confidence to the approximations used in this work for the Green’s function. Note that the Green’s function is independent of the thermal properties of the mine.

We next assess the ability of the model to describe the temperature over a simulated circular anti-tank mine buried in soil. The simulant mine has a diameter of 20 cm and a height of 7.5 cm. The mine is assumed to be a perfect insulator. Figure 4 shows the surface temperature over the center on the mine as a function time. Again, good agreement is noted, which provides a level of confidence that these disparate methods are both producing correct results.
Figure 3. The surface temperature over homogeneous soil as a function of time using an analytical solution, the integral equation solution described herein, and a FEM solution.

Figure 4. The surface temperature distribution over the center of a cylindrical time using the integral equation solution and a finite element solution.

7. SUMMARY AND CONCLUDING REMARKS

An integral equation is formulated and solved for the temperature over a buried land mine. Periodic (diurnal) solar heating is assumed, which makes possible a Fourier decomposition of the time variation. The integral equation for each Fourier harmonic of the temperature distribution is solved using the method of weighted residuals. The resulting model provides for a relatively simple geometry a reference solution, which can be used to check more sophisticated FEM-based models.
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REFERENCES


