# Density Matrices and Optical Bloch Equations

Assigned: March 25, 2020
Due: April 1, 2020

**Problem 1:** The Hamiltonian for an open system consisting of a two-level system interacting with a classical field and a bath is given by the Lindblad master equation:

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \sum_i \gamma_i \left[ L_i \rho L_i^\dagger - \frac{1}{2}\left( L_i^\dagger L_i \rho + \rho L_i^\dagger L_i \right) \right] \tag{1}$$

The system-bath interaction causes the spontaneous decay of the atoms excited state to the ground state. In this simple case there is only one jump operator $L$ to consider, with a rate $\gamma$.

**(a)** What is the correct jump operator $L$ needed to represent spontaneous decay? Show why this is correct.

**(b)** Using

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} \tag{2}$$

write down the spontaneous decay part of the master equation in matrix form. Explain why it makes sense.

**(c)** We will use the (by now) familiar two-level system Hamiltonian

$$H = \frac{\hbar}{2} \begin{pmatrix} 2\omega_0 & \Omega e^{-i\omega t} \\ \Omega^* e^{i\omega t} & 0 \end{pmatrix} \tag{3}$$

with $\Omega$ as the Rabi frequency, which depends on the applied electric field amplitude, phase, and the atom's dipole moment, and $\omega_0$ as the energy level of the excited state. Derive the equations of motion for the density matrix components $\rho_{ij}$. You do not need to solve them, but you do need to make a transformation which eliminates explicit time dependence from the equations of motion. Write your solutions in terms of $\delta = \omega - \omega_0$, $\gamma$, $\Omega$, and $\Omega^*$.

These equations are known as the Optical Bloch Equations (OBE).

**(d)** For a two-level system, the rate $R$ at which photons from the applied field are scattered by the atom is given by the rate at which the atom would jump from an excited to ground state ($\gamma$) multiplied by the probability of being in that excited state. Use the OBE to find the steady-state scattering rate $R$ for a two-level system. Write your solution in terms of $\delta$, $\gamma$, and $s_0 \equiv 2|\Omega|^2/\gamma^2$. Here $s_0$ is called the *on-resonance saturation parameter*, and it also often denoted by $s_0 = I/I_{\text{sat}}$, where $I$ is the intensity of the applied field, and $I_{\text{sat}}$ is known as the *saturation intensity*.

**(e)** Plot the scattering rate $R$ as a function of $\delta$ for $\Omega = \{0.3, 1.0, 3.0, 10\}$. You should notice two features: saturation, and power broadening. Explain in your own words what you see, and why these terms make sense. What is the maximum steady-state scattering rate for a fixed $\gamma$ and $\delta$?

**(f)** QuTiP's `mesolve` function allows for a simple implementation of the master equation using the `c_ops` input to set the jump operators. Write code to numerically solve the master equation for an open two-level system. Plot the excited state probability vs $\gamma t$, assuming you start in the ground state and $\delta = -\Omega = -\gamma$.