Perturbation Theory

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Problem 1: In the framework of non-degenerate perturbation theory, calculate the second order (proportional to λ^2) change to state $|n\rangle_\lambda$, as well as the third order (proportional to λ^3) energy shift. Using the notation from the lectures, this means calculating $|n^{(2)}\rangle$ and $E_n^{(3)}$.

Problem 2: Consider the 1D harmonic oscillator problem with a quartic perturbation. This Hamiltonian for a particle of mass m can be written as:

$$H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} + \lambda\hbar\omega\left(\frac{\hat{x}}{d}\right)^4 \tag{1}$$

where $\hat{x} = d(\hat{a}^\dagger + \hat{a})/\sqrt{2}$ and $\hat{p} = i\hbar(\hat{a}^\dagger - \hat{a})/\sqrt{2}d$ are the position and momentum operators, and $d = \sqrt{\hbar/m\omega}$ is the characteristic length scale of the harmonic oscillator. Here we are using λ as the dimensionless scaling parameter to keep track of the perturbations.

(a) Using non-degenerate perturbation theory, calculate the energy shifts of the ground state $|0\rangle$ up to order λ^3.

(b) This problem is actually a very important topic in quantum mechanics with a long history of being studied, resulting in hundreds of published papers. It has no known analytic solutions. As far back as 1969, Bender and Wu (Phys. Rev., Vol. 184, No. 5, p. 1231-1260) actually calculated the energy shift of the ground state to several more orders in the perturbation. Skim this paper to see the higher order terms. What, if any, is the radius of convergence? In other words, what, if any, is the minimum value of λ which guarantees convergence?

(c) This is known as an asymptotic expansion. Plot or tabulate the ground state energy shifts $E_0^{(k)}$ vs k to a sufficiently high order in λ for the following three values of λ: 0.2, 0.1, and 0.05. Use this information to estimate the total energy shift of the ground state for each value of λ.

(d) Confirm your results by using QuTiP to numerically calculate the ground state energy directly. Compare your estimates from Part (c) to the numerical solutions.

Problem 3: Let's now consider a degenerate case: The 2D harmonic oscillator with the following Hamiltonian:

$$H = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{m\omega^2}{2}(\hat{x}^2 + \hat{y}^2) + \lambda m\omega^2\hat{x}\hat{y} \tag{2}$$

where we have a perturbative term, denoted as usual with λ, which couples the position operators of the two dimensions.

(a) This problem is analytically solvable. Re-write this Hamiltonian in terms of $\hat{x} + \hat{y}$ and $\hat{x} - \hat{y}$. Then, choose a new canonical $\widehat{X}$, $\widehat{P}_x$, $\widehat{Y}$, and $\widehat{P}_y$ which obey the typical position-momentum commutation relations $[\widehat{X}, \widehat{P}_x] = [\widehat{Y}, \widehat{P}_y] = i\hbar$.

(b) It should now be clear what the eigenvalues of the Hamiltonian are. Write down the general solution, as well as specifically the three lowest energy states.

(c) Now, let's use perturbation theory. First, what is the expression for the unperturbed energy levels? How many degenerate states are there per energy level?

(d) Calculate the energy shift of the ground state to order λ^2, as well as the energy shift of the first and second excited states to order λ^1.

(e) Compare your perturbative solutions to the exact spectrum.

(f) Finally, use QuTiP to numerically calculate the energy levels of the three lowest energy states for $\lambda = 0.1$. Does it match the exact spectrum?