ECE 590.01 Quantum Engineering with Atoms

Spring 2020
Mid-Term Exam #1 – Revised 3/2/2020
Due on Friday 3/6/2020 12:00 pm EST

Instructor: Geert Vrijsen and Jungsang Kim

Name: ____

I hereby certify that I worked on this mid-term exam strictly abiding by the rules set forth by the instructor, as indicated below. I understand that the violation of these rules will be considered an academic misconduct, and will be subject to punishment. I also certify that I participated in the exam following the academic integrity and honesty anticipated for all Duke students.

Exam Rules:

1. I did not utilize any other documents, in paper, electronic or in any other format, other than the following allowed documents: the recommended textbook for the class (Cohen-Tannoudji, Diu and Laloe, Quantum Mechanics), classroom material provided through the website, notes I have taken in the classroom, and the homework problems.
2. I did not utilize any other resources to solve the exam problems, including any other textbooks or papers, homework/exam problems/solutions from similar classes taught in the past at Duke or elsewhere, or information available through web searches.
3. I worked on the problems by myself, and did not discuss the problems with anyone else other than the instructor; including my classmates, friends or colleagues, other professors, etc.

Signature ____ Date____________

1. Explain the following concepts:

a. What is a central potential, and what is its significance? (5 points)

b. What is Rydberg constant, and what does it represent? (5 points)

c. Can you explain why the hyperfine splitting (the energy difference) of the ground state of the hydrogen atom is so stable? (5 points)

d. What situation does Fermi's golden rule describe? (5 points)

2. Angular Momentum

a. From the fundamental commutation relation $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$, prove the following relations where *r* is the radial coordinate in spherical coordinates (10 points)

$$[\hat{L}_i, \hat{x}_j] = i\hbar\varepsilon_{ijk}\hat{x}_k,\ [\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k, \text{ and } [\hat{L}_i, f(r)] = 0.$$

b. For the eigenstate of the $\hat{L}^2$ and the $\hat{L}_z$ operator $|l = 1, m = 1\rangle$, find the probability that the measurement of $\hat{L}_x$ would yield the values $\hbar$, 0 and $-\hbar$. (10 points).

3. **Particle in a central potential**

A quantum particle is in a central potential, and the state is given by

$\psi(\vec{r}) = A_0(xy + yz + zx)e^{-\alpha r^2}$. You can use the table in the next page to answer these questions.

a. What is the probability to find $\hat{L}^2$ value of 0 and $6\hbar^2$, respectively? (10 points)

b. What is the probability to find the $\hat{L}_z$ value of $-2\hbar, -\hbar, 0, \hbar,$ and $2\hbar$, respectively? (10 points)

Spherical Harmonics

$l = 0$

$$Y_0^0(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{1}{\pi}}$$

$l = 1$

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot \frac{(x-iy)}{r}$$

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos\theta = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \frac{z}{r}$$

$$Y_1^{1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot \frac{(x+iy)}{r}$$

$l = 2$

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x-iy)^2}{r^2}$$

$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot \cos\theta = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x-iy)z}{r^2}$$

$$Y_2^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot (3\cos^2\theta - 1) = \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot \frac{(2z^2 - x^2 - y^2)}{r^2}$$

$$Y_2^{1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot \cos\theta = -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x+iy)z}{r^2}$$

$$Y_2^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot \frac{(x+iy)^2}{r^2}$$

Recurrence Relation for Associated Laguerre Polynomials

$$\rho L_p^q(\rho) = (2p+q+1)L_p^q(\rho) - \left[\frac{p+1}{p+q+1}\right] L_{p+1}^q(\rho) - (p+q)^2 L_{p-1}^q(\rho)$$

Orthonormality Condition for Electronic States of a Hydrogen Atom

$$\langle \varphi_{n'l'm'} | \varphi_{nlm} \rangle = \delta_{n'n}\delta_{l'l}\delta_{m'm}$$

4. Rydberg Atoms

The atoms excited to a high angular momentum state is called a Rydberg atom. Here, we explore the properties of the hydrogen atoms excited to the Rydberg state.

a. Show that for a hydrogen atom in the state corresponding to maximum orbital angular momentum ($l = n - 1$), (10 points)

$$\langle n, n-1|r|n, n-1\rangle = a_0 n\left(n+\frac{1}{2}\right)$$

$$\langle n, n-1|r^2|n, n-1\rangle = a_0^2 n^2 (n+1)\left(n+\frac{1}{2}\right)$$

b. Under the same conditions as in problem 4a, show that for large values of n and l, (10 points)

$$\sqrt{\langle \hat{r}^2 \rangle} \to a_0 n^2$$
$$\frac{\Delta r}{\langle r \rangle} \to 0$$
$$E_n \to -\frac{1}{2}\frac{e^2}{n^2 a_0}$$

where $(\Delta r)^2 = \langle \hat{r}^2 \rangle - \langle \hat{r} \rangle^2$.

5. Interacting spin systems (Cohen-Tannoudji, Liu and Laloe Exercise 2 in E$_{XIII}$)
In this problem, we consider two spin ½ particles coupled by an interaction Hamiltonian of the form $a(t)\vec{S}_1 \cdot \vec{S}_2$, where $a(t)$ approaches zero when $|t|$ approaches infinity, and takes on a non-negligible value on the order of a_0 only within the time interval $|t| \leq \tau$ near $t = 0$.

a. At $t = -\infty$, the system is in the state $|\uparrow\downarrow\rangle$, where the first spin is in the + eigenstate of $\hat{S}_{1z}$ and the second spin is in the – eigenstate of $\hat{S}_{2z}$. Without any approximation, calculate the state of the system at $t = +\infty$. Show that the transition probability $\mathcal{P}(\uparrow\downarrow \rightarrow \downarrow\uparrow)$ of finding the system in the $|\downarrow\uparrow\rangle$ state at $t = +\infty$ depends only on the integral $\int_{-\infty}^{\infty} a(t)dt$. (10 points)

b. Calculate $\mathcal{P}(\uparrow\downarrow \rightarrow \downarrow\uparrow)$ by using first-order time-dependent perturbation theory. Discuss the validity conditions for such an approximation by comparing the results with that obtained in part a. (10 points)