Chapter 3

Magnetic Moment

3.1 Magnetic Moment of an Electron

3.1.1 Spin 1/2 system and Pauli spin matrices

The eigenvectors of an angular momentum operator corresponding to a given eigenvalue j forms a basis for a vector space. The smallest such vector space is spanned when $j = 0$, but it is a single state and is not very interesting. The smallest non-trivial vector space corresponding to an angular momentum operator is formed when $j = 1/2$. Let's explore the structure of the algebra in this vector space. For convenience, we will use the notation $\hat{\boldsymbol{S}}$ to denote this section, to describe *the spin*, an intrinsic internal degree of freedom for a quantum elementary particle, that behaves exactly like an angular momentum when it comes to quantum operators. The quantum number for spin can take any integer or half-integer values depending on the elementary particle, but we first consider the case where s, the quantum number corresponding to $\hat{\boldsymbol{S}}^2$, is 1/2.

For this space, the components of the spin angular momentum operator $\hat{\boldsymbol{S}}$ satisfy the commutation relation

$$[\hat{S}_i, \hat{S}_j] = i\hbar\varepsilon_{ijk}\hat{S}_k, \tag{3.1}$$

just like the components of an angular momentum operator discussed in Chapter 1. The eigen-equation is given by

$$\begin{aligned} \hat{\boldsymbol{S}}^2|s, m_s\rangle &= \hbar^2 s(s+1)|s, m_s\rangle, \\ \hat{S}_z|s, m_s\rangle &= \hbar m_s|s, m_s\rangle \end{aligned} \tag{3.2}$$

For the case when spin $s = 1/2$, we have two eigenstates that correspond to m_s eigenvalues of 1/2 and $-1/2$, which we denote as $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively, that satisfies

$$
\begin{aligned}
\hat{\boldsymbol{S}}^2|\uparrow\rangle &= \tfrac{3}{4}\hbar^2|\uparrow\rangle, & \hat{S}_z|\uparrow\rangle &= \tfrac{\hbar}{2}|\uparrow\rangle, \\
\hat{\boldsymbol{S}}^2|\downarrow\rangle &= \tfrac{3}{4}\hbar^2|\downarrow\rangle, & \hat{S}_z|\downarrow\rangle &= -\tfrac{\hbar}{2}|\downarrow\rangle.
\end{aligned} \tag{3.3}
$$

Also, from the condition

$$
\hat{S}_\pm|s,m_s\rangle = \hbar\sqrt{s(s+1)-m_s(m_s\pm 1)}|s,m_s\pm 1\rangle = \hbar\sqrt{(s\mp m_s)(s\pm m_s+1)}|s,m_s\pm 1\rangle, \tag{3.4}
$$

we derive

$$
\begin{aligned}
\hat{S}_+|\uparrow\rangle &= 0, & \hat{S}_-|\uparrow\rangle &= \hbar|\downarrow\rangle, \\
\hat{S}_+|\downarrow\rangle &= \hbar|\uparrow\rangle, & \hat{S}_-|\downarrow\rangle &= 0.
\end{aligned} \tag{3.5}
$$

Noting that $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$, we deduce

$$
\begin{aligned}
\hat{S}_x|\uparrow\rangle &= \tfrac{\hbar}{2}|\downarrow\rangle, & \hat{S}_x|\downarrow\rangle &= \tfrac{\hbar}{2}|\uparrow\rangle, \\
\hat{S}_y|\uparrow\rangle &= -\tfrac{\hbar}{2i}|\downarrow\rangle, & \hat{S}_y|\downarrow\rangle &= \tfrac{\hbar}{2i}|\uparrow\rangle,
\end{aligned} \tag{3.6}
$$

From here, we can find the matrix representation of the $\hat{S}_i$ operators in this basis as

$$
\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \hat{S}_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{3.7}
$$

Pauli matrices are defined as $\hat{\boldsymbol{\sigma}} \equiv (2/\hbar)\hat{\boldsymbol{S}}$, and therefore

$$
\hat{\sigma}_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \equiv \hat{X}, \quad \hat{\sigma}_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \equiv \hat{Y}, \quad \hat{\sigma}_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \equiv \hat{Z}. \tag{3.8}
$$

Along with the 2×2 identity operator, Pauli operators form the basis for all operators acting on the Hilbert space of the $s = 1/2$ spin states.

3.1.2 Properties of Pauli matrices

The commutation relationship for Pauli matrices can be derived from the commutation relation of the spin operators in Eq. 3.1 as

$$
[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\varepsilon_{ijk}\hat{\sigma}_k. \tag{3.9}
$$

One can readily verify the anti-commutation relation

$$
\{\hat{\sigma}_i, \hat{\sigma}_j\} \equiv \hat{\sigma}_i\hat{\sigma}_j + \hat{\sigma}_j\hat{\sigma}_i = 2\hat{I}\delta_{ij}. \tag{3.10}
$$

We can also verify

$$
\hat{\sigma}_i^2 = \hat{I}, \quad \hat{\sigma}_i\hat{\sigma}_j = i\varepsilon_{ijk}\hat{\sigma}_k \quad \longrightarrow \quad \hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\hat{I} + i\varepsilon_{ijk}\hat{\sigma}_k. \tag{3.11}
$$

The x and y eigenstates of the Pauli matrices are given by

$$\hat{\sigma}_x|\pm\rangle = \pm|\pm\rangle, \quad |\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle), \tag{3.12}$$

$$\hat{\sigma}_y|\pm i\rangle = \pm|\pm i\rangle, \quad |\pm i\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm i|\downarrow\rangle). \tag{3.13}$$

The z eigenstates are $|\uparrow\rangle$ and $|\downarrow\rangle$, as shown in Eq. 3.3.

3.1.3 Free particle wavefunctions including spin

Since spin is a completely independent, internal degree of freedom, the spin operators $\hat{\boldsymbol{S}}^2$ and $\hat{S}_z$, which commutes with each other, also commute with the position operators ($\hat{x}$, $\hat{y}$, $\hat{z}$). Therefore, the interaction of spin with the position or the momentum operator is only introduced in the presence of electromagnetic field. For a free particle, the spin degree of freedom separates from the spatial degree of freedom. For a given spatial wavefunction solution to Schrödinger's equation $\phi_k(\vec{r})$, there are two different solutions $\varphi_+ \equiv \phi_k(\vec{r})|\uparrow\rangle$ and $\varphi_- \equiv \phi_k(\vec{r})|\downarrow\rangle$, which satisfy the equations

$$\hat{H}\varphi_+ = \frac{\hbar^2 k^2}{2m}\varphi_+, \quad \hat{\boldsymbol{S}}^2\varphi_+ = \frac{3}{4}\hbar^2\varphi_+, \quad \hat{S}_z\varphi_+ = \frac{\hbar}{2}\varphi_+, \tag{3.14}$$

$$\hat{H}\varphi_- = \frac{\hbar^2 k^2}{2m}\varphi_-, \quad \hat{\boldsymbol{S}}^2\varphi_- = \frac{3}{4}\hbar^2\varphi_-, \quad \hat{S}_z\varphi_- = -\frac{\hbar}{2}\varphi_-. \tag{3.15}$$

For a given energy E_k, there are four possible solutions with the same energy (degeneracy), $\varphi_+(+\vec{k})$, $\varphi_+(-\vec{k})$, $\varphi_-(+\vec{k})$ and $\varphi_-(-\vec{k})$.

3.1.4 Electron magnetic moment

Figure 3.1 schematically shows the definition of a magnetic dipole moment vector $\vec{m}$ generated by a loop of current I enclosing the area A. One can keep a finite magnetic dipole moment in the limit the area of the loop shrinks to zero, if the current increases proportionately to keep the product constant. Such magnetic dipole moment is defined as

$$\vec{m} = \frac{I}{2}\oint_{loop} \vec{r} \times d\vec{s} = \frac{1}{2}\int_V \vec{r} \times \vec{J}(\vec{r})d\tau, \tag{3.16}$$

where $\vec{J}(\vec{r})$ is the current density at position $\vec{r}$. The Biot-Savart law relates the magnetic flux density $\vec{B}$ to the current density $\vec{J}(\vec{r})$ as

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \vec{J}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}d\vec{r}' = \frac{\mu_0}{4\pi}\oint_{loop} \frac{Id\vec{s}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}, \tag{3.17}$$

where $\mu_0 = 4\pi \times 10^{-7}$H/m is magnetic permeability of vacuum. we note that the coordinate $\vec{r}$ is where we measure the magnetic flux density, and $\vec{r}'$ is the coordinate o fthe current source.

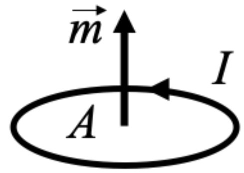

Figure 3.1: Model for a magnetic moment. When current I flows in a loop of an area A, the magnetic moment vector $\vec{m}$ takes a magnitude of IA and direction perpendicular to the area following the right hand rule.

We consider $\vec{R} = \vec{r} - \vec{r}'$, its magnitude $R = |\vec{r} - \vec{r}'|$ and its gradient

$$\nabla\left(\frac{1}{R}\right) = -\frac{1}{R^2}\nabla R = -\frac{\vec{R}}{R^3}, \tag{3.18}$$

to note that

$$d\vec{s}' \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = -d\vec{s}' \times \nabla\left(\frac{1}{R}\right) = \nabla \times \left(\frac{d\vec{s}'}{R}\right) - \frac{1}{R}\nabla \times d\vec{s}'. \tag{3.19}$$

The second term on the right hand side of Eq. 3.19 is zero, as the gradient is taken over the coordinate system r. Using this relationship, Eq. 3.17 simplifies to

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\oint_{loop} \nabla \times \left(\frac{Id\vec{s}'}{R}\right) = \nabla \times \left(\frac{\mu_0}{4\pi}\oint_{loop}\frac{Id\vec{s}'}{R}\right). \tag{3.20}$$

Using the definition of vector potential $\vec{B} = \nabla \times \vec{A}$, we recognize the vector potential as

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint_{loop}\frac{d\vec{s}'}{R}. \tag{3.21}$$

In the limit the current loop is small compared to the distance between the loop and the location where the magnetic flux density is being measured ($|\vec{r}'| \ll |\vec{r}|$), the coordinate $1/R$ can be expanded as

$$\frac{1}{R} = \frac{1}{r}\frac{1}{\left[1 - 2\frac{\vec{r}\cdot\vec{r}'}{r^2} + \frac{r'^2}{r^2}\right]^{1/2}} \approx \frac{1}{r}\left[1 + \frac{\vec{r}\cdot\vec{r}'}{r^2} - \frac{r'^2}{2r^2}\right] = \frac{1}{r}\left[1 + \frac{\hat{r}\cdot\vec{r}'}{r} - \frac{r'^2}{2r^2}\right], \tag{3.22}$$

which is also known as the multipole expansion, and $\hat{r} = \vec{r}/r$ is the unit vector along the direction of $\vec{r}$. The first term is the magnetic monopole term, where $\vec{A}_M \propto \oint d\vec{s}' = 0$ indicates that this term vanishes. The first non-zero term is the second term in the expansion known as the dipole. For this term, we note that

$$(\hat{r}\cdot\hat{r}')d\vec{s}' = \vec{r}'(\hat{r}\cdot d\vec{s}') - \hat{r} \times (\vec{r}' \times d\vec{s}') \tag{3.23}$$

to derive the dipole term (the first term on the right hand side integrates to zero due to symmetry)

$$\vec{A}_D = \frac{\mu_0 I}{4\pi r^2}\oint_{loop}(\hat{r}\cdot\hat{r}')d\vec{s}' = \frac{\mu_0}{4\pi r^2}\left[\frac{I}{2}\oint_{loop}\hat{r}' \times d\vec{s}'\right] \times \hat{r} = \frac{\mu_0}{4\pi r^3}\vec{\mu} \times \vec{r}. \tag{3.24}$$

For charge q moving around the loop with velocity $\vec{v}$, we note that $Id\vec{s} = qd\vec{v}$ to see that

$$\vec{m} = \frac{I}{2}\oint_{loop} \vec{r} \times d\vec{s} = \frac{q}{2}\oint_{loop} \vec{r} \times d\vec{v} = \frac{q}{2m}\oint_{loop} \vec{r} \times d\vec{p} = \frac{q}{2m}\oint_{loop} d\vec{L} = \frac{q}{2m}\vec{L}. \tag{3.25}$$

The coefficient relating the magnetic moment to the angular momentum, namely the ratio of the charge and twice the particle mass ($q/2m$), is called the gyromagnetic ratio. For an elementary charge of $q = -e$, electron mass m_e and a unit angular momentum corresponding to the Planck Constant $\hbar$, the unit of magnetic moment becomes

$$\vec{\mu} = -\frac{e\hbar}{2m_e}\vec{\sigma} = -\mu_B\vec{\sigma}, \tag{3.26}$$

where $\mu_B = 9.27 \times 10^{-24}\text{Am}^2$ is called the Bohr magneton, an elementary unit of magnetic moment.

For an electron in orbit, the corresponding orbital magnetic moment is given by $\vec{m}_{orb} = -e\vec{L}/2m_e$. For the internal angular momentum given by the spin, we use $2\vec{S}$ as its angular momentum, so that the total magnetic moment is given by

$$\vec{m} = -\frac{e}{2m_e}(\vec{L} + 2\vec{S}). \tag{3.27}$$

It is useful to note that the total angular momentum of an orbiting electron indeed is $\vec{J} = \vec{L} + \vec{S}$. When we relate the total angular momentum $\vec{J}$ of the electron to the magnetic moment, we have to introduce a proportionality constant g;

$$\vec{m} = -g\frac{e}{2m_e}\vec{J}, \tag{3.28}$$

and g is known as the Landé g-factor.

For the case when the orbital angular momentum is zero, we get

$$\vec{m} = -\frac{e}{m_e}\vec{S} = -\frac{e\hbar}{2m_e}s\vec{\sigma} = -\mu_B\vec{\sigma}, \tag{3.29}$$

so we see that the intrinsic magnetic moment of an electron has a magnitude that equals Bohr magneton.

3.2 Spins in Magnetic Field

3.2.1 Force on electron spins

The electron spin is an internal degree of freedom, but if it has a phyical meaning (*i.e.* if we want to associate such a degree of freedom), this must be measurable using an experimental scheme. Such an experiment was performed by Otto Stern and Walther Gerlach in 1922, known as the Stern-Gerlach experiment. In this experiment, the electron beam is passed through a non-uniform magnetic field with a strong gradient. In the presence of a magnetic field, there is a torque acting on the magnetic moment $\vec{N} = \vec{\mu} \times \vec{B}$ in the direction that tends to align the magnetic moment with the magnetic field. The potential energy in this case is given by $V = -\vec{\mu} \cdot \vec{B}$. The net force on the particle is given by

$$F = -\nabla V = \nabla(\vec{\mu} \cdot \vec{B}) = -(\vec{\mu} \cdot \nabla)\vec{B} - (\vec{B} \cdot \nabla)\vec{\mu} - \vec{\mu} \times (\nabla \times \vec{B}) - \vec{B} \times (\nabla \times \vec{\mu}). \tag{3.30}$$

The second and the fourth term vanishes, as the magnetic moment $\vec{\mu}$ is a point property of the particle that does not depend on the position of the particle. The third term can be reduced to $\sim \vec{\mu} \times \partial\vec{E}/\partial t$ due to Maxwell's equation (Ampère's law), and can be ignored since the magnetic moment does not interact with the electric field. So, the electron magnetic moment (due to internal spin) interacts with the gradient of the magnetic field. Indeed, the Stern-Gerlach experiment revealed that the stream of electrons are split into two parts, indicating that there are two different states of the electron even if they occupy the same spatial location.

3.2.2 Precession of an electron in a magnetic field

Consider an electron fixed in space, in the presence of a constant magnetic field in z-direction, $\vec{B} = B\hat{z}$. Even in the absence of orbital angular momentum, the internal degree of freedom (spin) of the electron interacts with the magnetic field. We consider the case where the initial spin state at $t = 0$ is in an equal superposition of the two eigenstates of the $\hat{S}_z$ operator

$$\xi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.31}$$

The Schrödinger equation reads

$$i\hbar\frac{\partial}{\partial t}\xi(t) = \hat{H}\xi(t), \tag{3.32}$$

where $\hat{H} = \vec{\mu} \cdot \vec{B} = \mu_B \vec{\sigma} \cdot \vec{B} = \mu_B B \hat{\sigma}_z$. At time t using

$$\xi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}, \tag{3.33}$$

the Schrödinger equation becomes

$$\begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix} = -\frac{i\Omega}{2} \begin{pmatrix} a(t) \\ -b(t) \end{pmatrix}, \tag{3.34}$$

where $\Omega/2 \equiv |e|B/2m$ is called the Larmor frequency. The solution to this equation is

$$\xi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\Omega t/2} \\ e^{i\Omega t/2} \end{pmatrix}. \tag{3.35}$$

On a Bloch sphere where the two eigenstates represent the north and the south pole, this solution indicates rotation on the equator with a frequency of Ω.

We can also solve for the eigenenergy of the system, by setting up a time-independent Schrödinger equation $\hat{H}\xi = E\xi$. In matrix representation using the $\hat{\sigma}_z$ basis (using the two eigenstates of the $\hat{\sigma}_z$ operator as the basis states), we get

$$\mu_B B \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = E \begin{pmatrix} a \\ b \end{pmatrix}. \tag{3.36}$$

Letting $\lambda = E/\mu_B B$, we find the secular equation to be $(1-\lambda)(1+\lambda) = 0$, yielding two solutions $\lambda = \pm 1$. The two eigen energy solutions are $E = \pm\mu_B B = \pm\hbar\Omega/2$, and the eigenstates are $\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for $E = \hbar\Omega/2$ and $E = -\hbar\Omega/2$, respectively.

3.2.3 Magnetic resonance

We have seen that for static electrons (with zero orbital angular momentum), the magnetic moment is given by $\vec{\mu} = g\mu_B\vec{S}/\hbar$, where the Landé g-factor is $g = 2$. For proton or neutron spin, we define the nuclear magneton

$$\mu_N = \frac{e\hbar}{2M_p} = 5.05 \times 10^{-27}\mathrm{Am}^2. \tag{3.37}$$

The nuclear magnetic moment is given by $\vec{\mu} = g\mu_N\vec{S}/\hbar$, where the Landé g-factor for proton is $g_p = s \times 2.79$ and that for neutron is $g_n = 2 \times (-1.91)$.

The nuclear magnetic moment can be measured very accurately using a magnetic resonance technique (called nuclear magnetic resonance, or NMR). We have seen in the previous section that when the magnetic field $\vec{B}$ is parallel to the spin state $\vec{S}$, the two spin states are eigenstates and the energy is conserved. On the other hand, when the $\vec{B}$-field is perpendicular to the spin state, then the spin state precesses. We need both components to characterize the magnetic moment. Consider a uniform magnetic field along a general direction $\vec{B} = (B_x, B_y, B_z)$, and a general spin state $\xi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$. The time-dependent Schrödinger equation reads

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = -\frac{g\mu_N}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} B_y + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} B_z\right]\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}. \tag{3.38}$$

Now consider rotating the perpendicular component $B_\perp$ of the magnetic field at a frequency of ω such that

$$B_x = B_\perp \cos\omega t, \quad B_y = -B_\perp \sin\omega t, \quad B_z = B_\parallel, \tag{3.39}$$

to obtain two differential equations

$$\frac{\partial}{\partial t}a(t) = i\left(\Omega_\perp b(t)e^{i\omega t} + \Omega_\parallel a(t)\right), \tag{3.40}$$

$$\frac{\partial}{\partial t}b(t) = i\left(\Omega_\perp a(t)e^{-i\omega t} - \Omega_\parallel b(t)\right), \tag{3.41}$$

where $\Omega_\perp = g\mu_N B_\perp/2\hbar$ and $\Omega_\parallel = g\mu_N B_\parallel/2\hbar$ are the Larmor frequencies corresponding to the perpendicular and parallel components of the magnetic field, respectively. We consider the case where the spin is initialized in the $|\uparrow\rangle$ state, such that $a(t=0)=1$ and $b(t=0)=0$. We conjecture that the solution will have an oscillating form, *i.e.*, $a(t)=\bar{a}e^{i\omega_a t}$ and $b(t)=\bar{b}e^{i\omega_b t}$. Then, the differential equations 3.40 and 3.41 reduces to a pair of linear equations that can be expressed as

$$\begin{pmatrix} \omega_a - \Omega_\parallel & -\Omega_\perp e^{i(\omega+\omega_b-\omega_a)t} \\ -\Omega_\perp e^{i(-\omega+\omega_a-\omega_b)t} & \omega_b + \Omega_\parallel \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = 0. \tag{3.42}$$

If we define $\phi \equiv \omega_a - \omega_b - \omega$, we note that the two equations have to be independent of time, *i.e.*, $\phi = 0$. This leads to the condition that $\omega_a = \omega_b + \omega$. In order for Eq. 3.42 to have a non-trivial solution, the characteristic polynomial of the matrix must be zero;

$$\left(\omega_b + \omega - \Omega_\parallel\right)\left(\omega_b + \Omega_\parallel\right) - \Omega_\perp^2 = 0. \tag{3.43}$$

The solution to this equation is $\omega_b = -\omega/2 \pm \bar{\omega}$ and $\omega_a = \omega_b + \omega = \omega/2 \pm \bar{\omega}$, where

$$\bar{\omega} = \sqrt{\left(\frac{\omega}{2}\right)^2 + \left(\Omega_\parallel^2 + \Omega_\perp^2 - \Omega_\parallel \omega\right)}, \tag{3.44}$$

or,

$$\bar{\omega}^2 = \left(\Omega_\parallel - \frac{\omega}{2}\right)^2 + \Omega_\perp^2. \tag{3.45}$$

The solution for $b(t)$ is a linear superposition of the two possible solutions

$$b(t) = b_1 e^{-i(\omega/2-\bar{\omega})t} + b_2 e^{-i(\omega/2+\bar{\omega})t}. \tag{3.46}$$

The initial solution $b(t=0) = b_1 + b_2 = 0$ indicates that $b_1 = -b_2 \equiv c/(2i)$, which leads to

$$b(t) = \frac{c}{2i}\left[e^{-i(\omega/2-\bar{\omega})t} - e^{-i(\omega/2+\bar{\omega})t}\right] = c e^{-i\omega t/2}\sin\bar{\omega}t. \tag{3.47}$$

Taking the time derivative and inserting into Eq. 3.41 gives

$$\frac{\partial b}{\partial t} = -\frac{ic\omega}{2}e^{-i\omega t/2}\sin\bar{\omega}t - c\bar{\omega}\cos\bar{\omega}t = i\Omega_\perp a(t)e^{-i\omega t} - \Omega_\parallel c e^{-i\omega t/2}\sin\bar{\omega}t. \tag{3.48}$$

At $t=0$, we get $-c\bar{\omega} = i\Omega_\perp a(t=0)$, and noting that $a(t=0)=1$, we conclude $c = -i\Omega_\perp/\bar{\omega}$. Solving Eq. 3.48 for $a(t)$ yields

$$a(t) = -\frac{\sin\bar{\omega}t}{\bar{\omega}}\left[i(\Omega_\parallel - \frac{\omega}{2}) - \bar{\omega}\cot\bar{\omega}t\right]e^{i\omega t/2}. \tag{3.49}$$

So, the final solution for the state of the spin is given by $|\xi(t)\rangle = a(t)|\uparrow\rangle + b(t)|\downarrow\rangle$. One can easily check that this solution is properly normalized, *i.e.*, $|a(t)|^2 + |b(t)|^2 = 1$.

The probability of finding the spin in the $|\downarrow\rangle$ at a time t is given by

$$|b(t)|^2 = \left[\frac{\Omega_\perp^2}{\bar{\omega}^2}\right]\sin^2\bar{\omega}t = \left[\frac{\Omega_\perp^2}{(\Omega_\parallel^2 - \omega/2)^2 + \Omega_\perp^2}\right]\sin^2\bar{\omega}t, \tag{3.50}$$

and oscillates in time as $\sim \sin^2\bar{\omega}t$. The amplitude of the oscillation is maximized when the resonance condition $\omega = 2\Omega_\parallel = g\mu_N B_\parallel/\hbar$ is satisfied, where $\bar{\omega} = \Omega_\perp$. As you go off resonance, the oscillation frequency $\bar{\omega}$ increases, and the contrast $\Omega_\perp^2/\bar{\omega}^2$ decreases.