Chapter 12

Quantum Networking with Atomic Qubits

Classical communication networks are ubiquitous today, and widely used in the world for a variety of practical applications today. Communication networks are capable of sending classical information (bits) across long distances, and among a large number of communication nodes. A versatile communication network requires means to send bits over distances at various scales (local area networks within a building, campus area networks across buildings, campus or a neighborhood, metropolitan area networks covering towns or cities, and wide-area networks for a much wider range), and the ability to switch the communication traffic from the origin node to a destination node. Early communication networks were developed for telephony applications, but they are now widely used for data communications over the internet.

A similar function can be expected for quantum information, where qubits can be reliably communicated across a network. However, all communication networks operate over a lossy communication channel, and the information carrier often distorts or loses the signal on its way to its destination. Loss of optical photons in the transmission through an optical fiber, or loss of RF signal strength between a cell phone and the cell tower are classic examples of such channel losses. In classical communication channel, various strategies have been developed to protect the information being transmitted against such channel losses. Optical regenerators and optical amplifiers are examples where the attenuated signal is either measured and re-transmitted, or boosted to counteract the effect of signal reduction. Forward error correction is another strategy, where the information is represented by more than one signal carrier with redundancy, which can be used to digitally correct for the received information in case of signal loss or severe distortion. While these strategies are effective for classical information, they cannot readily be adopted for transmitting quantum information. This is because quantum information is subject to no-cloning theorem, where replicating qubits is not allowed, so amplification or redundant coding is not possible. Furthermore, a qubit in transit cannot be regenerated as a measurement of a qubit immediately leads to loss of quantum information.

Despite these challenges, construction of a quantum communication network is possible where qubits are transmitted over long distances (potentially transcontinental scale), and switched among multiple users in the network. In this chapter, we discuss the basic principles for constructing quantum networks, and how atomic qubits can be used to realize the basic elements that constitute such a network.

12.1 Quantum Teleportation with Bell States

12.1.1 Quantum teleportation

Quantum teleportation is a mechanism by which a maximally entangled qubit pair can be used to "ship" a qubit from one location to another, by sending two classical bits of information. We consider two parties Alice and Bob, each serving as a transmitter and receiver of a qubit #1, $|\psi\rangle_1 = \alpha|0\rangle_1 + \beta|1\rangle_1$, that Alice holds in the beginning. The actual quantum information is represented by the two comples numbers α and β. Alice and Bob shares a maximally entangled qubit pair, where Alice holds qubit #2 and Bob holds qubit #3. The maximally entangled state is represented by

$$|\psi_+\rangle_{23} = \frac{1}{\sqrt{2}}(|0\rangle_2|0\rangle_3 + |1\rangle_2|1\rangle_3). \tag{12.1}$$

Alice and Bob would use this entangled pair to send Alice's qubit state $|\psi\rangle_1$ to Bob. The combined state of the three qubit states can be expressed as

$$\begin{aligned}
|\psi\rangle_1|\psi_+\rangle_{23} &= \frac{1}{\sqrt{2}}(\alpha|0\rangle_1 + \beta|1\rangle_1)(|0\rangle_2|0\rangle_3 + |1\rangle_2|1\rangle_3) \\
&= \frac{1}{\sqrt{2}}(\alpha|0\rangle_1|0\rangle_2|0\rangle_3 + \alpha|0\rangle_1|1\rangle_2|1\rangle_3 + \beta|1\rangle_1|0\rangle_2|0\rangle_3 + \beta|1\rangle_1|1\rangle_2|1\rangle_3) \\
&= \frac{1}{2}[\alpha(|\psi_+\rangle_{12} + |\psi_-\rangle_{12})|0\rangle_3 + \alpha(|\phi_+\rangle_{12} + |\phi_-\rangle_{12})|1\rangle_3 \\
&\qquad + \beta(|\phi_+\rangle_{12} - |\phi_-\rangle_{12})|0\rangle_3 + \beta(|\psi_+\rangle_{12} - |\psi_-\rangle_{12})|1\rangle_3] \\
&= \frac{1}{2}[|\psi_+\rangle_{12}(\alpha|0\rangle_3 + \beta|1\rangle_3) + |\psi_-\rangle_{12}(\alpha|0\rangle_3 - \beta|1\rangle_3) \\
&\qquad + |\phi_+\rangle_{12}(\beta|0\rangle_3 + \alpha|1\rangle_3) - |\phi_-\rangle_{12}(\beta|0\rangle_3 - \alpha|1\rangle_3)],
\end{aligned} \tag{12.2}$$

where in moving to the third line, we used the definition of the four Bell states expressed in Eq. 11.2–11.5. The final expression in Eq. 12.2 consists of four terms, each showing one of four Bell states for the two qubits Alice has, that dictates the state for the qubit #3 that Bob has. Since the Alice has full access to the two qubits #1 and #2, and the four Bell states are orthonormal to each other, she can design a measurement to measure her two qubits in one of the four Bell states (this is called *measurement in Bell basis*). For example, she can apply a CNOT gate between her two qubits using qubit #1 as the control and qubit #2 as the target qubit, and then apply a Hadamard gate (Eq. 11.15) to qubit #1, and measure the two qubits in a computational basis. This procedure would transform Alice's two qubit states into

$$
\begin{aligned}
|\psi_+\rangle_{12} &= \frac{1}{\sqrt{2}}(|0\rangle_1 |0\rangle_2 + |1\rangle_1 |1\rangle_2) \quad \longrightarrow \quad |0\rangle_1 |0\rangle_2 \\
|\psi_-\rangle_{12} &= \frac{1}{\sqrt{2}}(|0\rangle_1 |0\rangle_2 - |1\rangle_1 |1\rangle_2) \quad \longrightarrow \quad |1\rangle_1 |0\rangle_2 \\
|\phi_+\rangle_{12} &= \frac{1}{\sqrt{2}}(|0\rangle_1 |1\rangle_2 + |1\rangle_1 |0\rangle_2) \quad \longrightarrow \quad |0\rangle_1 |1\rangle_2 \\
|\phi_-\rangle_{12} &= \frac{1}{\sqrt{2}}(|0\rangle_1 |1\rangle_2 - |1\rangle_1 |0\rangle_2) \quad \longrightarrow \quad |1\rangle_1 |1\rangle_2
\end{aligned}
$$

and the measurement in computational basis following this procedure would result in unambiguously distinguishing the four Bell states.

Depending on Alice's measurement outcome, Bob's qubit now falls into one of four possibilities. For example, when Alice's measurement outcome is 00, then Bob's state is in $\alpha |0\rangle_3 + \beta |1\rangle_3$, and reproduces Alice's original state. When Bob's measurement outcome is 10, then Bob's state is in $\alpha |0\rangle_3 - \beta |1\rangle_3$. In this case, Bob can apply a $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ gate to his qubit, to recover the original state $\alpha |0\rangle_3 + \beta |1\rangle_3$ that Alice started out with. Similarly, when Bob measures 01 (or 11), he can apply a $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (or $Y = -iZX$) gate to recover Alice's original qubit state on his qubit #3. This means that Alice will have to transmit two *classical* bits of information to Bob after her Bell state measurement to inform him of her measurement outcome, and Bob will choose one of the four operations $\{I, Z, X, -iZX\}$ and apply it to his qubit, depending on the two classical bits of information he receives from Alice. At the end of this process, Bob will unambiguously obtain $\alpha |0\rangle_3 + \beta |1\rangle_3$ state, and reproduces Alices original state without any errors, with unity probability. This process, where Alice can *transmit* her arbitrary qubit state to Bob using a maximally entangled pair that they share *a priori* by communicating two classical bits of information, is called **quantum teleportation**.

12.1.2 Properties of quantum teleportation

We should note a few key features in quantum teleportation. First, quantum teleportation is capable of sending an *arbitrary* qubit state from Alice to Bob. Alice does not necessarily know the state of her qubit, if it were handed to her by a third party. A different way to look at this is that Alice can teleport a state she has to Bob even if she does not know the exact quantum state is. Since she will only be sending two classical bits that correspond to her measurement output that do not reveal any information regarding α and β, the quantum teleportation process is **secure**. The classical information Alice sends to Bob can be made public, but no one other than Bob will be able to reproduce the original qubit state $\alpha |0\rangle + \beta |1\rangle$.

It is important to note that although Bob reproduces the original quantum state Alice had, the quantum teleportation procedure does not violate the quantum no-cloning theorem. This is because Alice's quantum state is destroyed as Alice completes her Bell state measurement. When the measurement outcome is transmitted to Bob as classical bits of information, the quantum state technically does not exist in Alice's hands any more. The original qubit state is only recovered after Bob applies his choice of single-qubit gate. At no time in this procedure does the original quantum state exists in more than one place at the same time.

If Alice has the ability to distinguish among all four Bell states, than this protocol has unit efficiency, *i.e.*, Bob can reliably reproduce Alice's state deterministically. However, if Alice can only identify parts of her Bell states, then quantum teleportation succeeds with only finite probability. For example, if Alice has a measurement scheme that can unmistakably identify one of the four Bell states, for example the $|\psi_+\rangle$ state, then her measurement will successfully yield this state 1/4 of the time and she knows when this measurement succeeds. The other 3/4 of the time, she knows that she did not get the $|\psi_+\rangle$ state, but cannot identify which one of the remaining three states she measured. In this case, she should give up, and the protocol fails. In this example, the teleportation only succeeds when she gets a measurement outcome consistent with the $|\psi_+\rangle$ state, with a probability of 1/4 (and Bob does not have to do anything to his qubit to recover the original qubit state. Sometimes, such a teleportation process with partial success is also helpful as long as the success is heralded (*i.e.*, both Alice and Bob unambiguously recognizes when the teleportation succeeds). Another example is when Alice can successfully detect two out of four Bell states, again, with heralded success. In such a case, the teleportation succeeds with a probability of 1/2.

12.1.3 Entanglement swapping

Now let's consider the case where Alice starts with two qubits, qubit #1 and qubit #4, where these two were originally prepared in a maximally entangled state themselves,

$$|\varphi\rangle_{14} = \frac{1}{\sqrt{2}}(|0\rangle_1 |0\rangle_4 + |1\rangle_1 |1\rangle_4). \tag{12.3}$$

Alice uses the quantum teleportation procedure to send the state of qubit #1 to Bob, who reproduces it in qubit #3. After the teleportation process, qubit #1 is destroyed, but the qubit #3 now holds the state qubit #1 was in before the teleportation process. It is straightforward to show that the quantum teleportation works even if qubit #1 was originally entangled with another qubit, namely qubit #4. After the teleportation, we end up with the state

$$|\varphi'\rangle_{34} = \frac{1}{\sqrt{2}}(|0\rangle_3 |0\rangle_4 + |1\rangle_3 |1\rangle_4), \tag{12.4}$$

where the state for qubit #1 is replaced with the same qubit state in qubit #3. We note that now Alice has qubit #4 and Bob has qubit #3, so they share a maximally entangled state.

In this example, the original entanglement was between qubits #1 and #4, and between #2 and #3. After the teleportation of qubit state #1 to qubit #3, we established an entanglement between qubit #1 and qubit #4, which have never directly interacted. This process of generating entanglement between two qubits that were each originally entangled with a different qubit, is called *entanglement swapping*, as the entanglement between qubit

#4 and qubit #1 is swapped to become an entanglement between qubit #4 and qubit #3. Entanglement swapping is a critical protocol to enable quantum networks. One note that in this example, none of the four qubits were prepared in any state other than a maximally entangled state, *i.e.*, no general coefficient α or β appears anywhere in the protocol. Entanglement swapping is a method to generate maximally entangled states between two qubits that never experiences a direct interaction.

12.2 Remote Entanglement Generation between Atomic Qubits

Next, we describe an experimental protocol where an entanglement can be generated between two atomic qubits without having them directly interact in a single trap, as in the examples of Cirac-Zoller or Mølmer-Sørensen gates. The key to this protocol is to use emission of photons from these atomic qubits, and let the photons interact. Here we first quickly summarize the properties of photon states.

12.2.1 Hong-Ou-Mandel effect: interference of identical bosons

In quantum mechanics, the quantization of the electromagnetic field leads to a conclusion that the energy of the optical field is quantized, and the unit of energy is proportional to the frequency of the optical field. This can be expressed by a "particle" of the optical field, called a *photon*, where each particle carries the unit energy of the field. The proportionality constant h that relates the unit energy and the frequency ν of the electromagnetic field $E = h\nu$ is called the Planck constant, and has a value $h = 6.626 \times 10^{-34}$Js. Each photon is characterized by its frequency ν, polarization, and the wavevector $\vec{k}$ corresponding to the spatial mode. The photon is derived by first solving the wave equation for the electromagnetic field using a given boundary solution. The solutions are the electromagnetic field modes specified by the spatial wavefunction (characterized by the wavevector $\vec{k}$, the polarization and the frequency). Once the mode solution is found, the photon model describes how the energy is distributed in each mode: that each mode can be excited by an integer multiple of the unit of energy corresponding to the energy of the photon.

Because the photon is a quantum mechanical particle, two photons with the same frequency, polarization and wavevector is *indistinguishable*, and when they interfere, you must assume that one cannot tell the identify of one photon from the other when they interact. This indistinguishability leads to two types of fundamental particles: one where the exchange of the particles leads to symmetric wavefunction (these particles are called bosons), and the other where the exchange of the particles leads to a negative sign and therefore antisymmetric wavefunction (these particles are called fermions). Mathematically, bosons are described by their creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ that obey a commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1, \tag{12.5}$$

and fermions are described by their creation and annihilation operators $\hat{c}^\dagger$ and $\hat{c}$ that obey an anti-commutation relation

$$\{\hat{c}, \hat{c}^\dagger\} = \hat{c}\hat{c}^\dagger + \hat{c}^\dagger\hat{c} = 1. \tag{12.6}$$

When two nominally identical classical particles enter a 50/50 beamsplitter and "collide", each input has 50% chance of being transmitted or reflected. This means that we have a 25% chance of both particles exiting each port of the beamsplitter, and 50% chance where one particles exits each port of the beamsplitter. The 50% probability consists of two different possibilities, 25% of which describes the case when both particles are transmitted at the beamsplitter, and the other 25 % of the case when both particles are reflected at the beamsplitter. This situation changes dramatically when the two particles are identical quantum particles.

When the two particles are indistinguishable quantum particles, one cannot distinguish the case where the two particles are both transmitted at the beamsplitter, and the case where they are both reflected at the beamsplitter. This is because the two incident particles and the two exiting particles are *fundamentally indistinguishable*, and one cannot, even in principle, track the interaction at the beamsplitter to see exactly which one of these two situations actually occurred. This means that the probability amplitudes of these two situations must be allowed to interfere (*i.e.*, the probability amplitudes have to be added up first), before the probabilities of the output scenarios can be evaluated.

When the two incident particles are bosons, the probability amplitudes for both particles being reflected carries a negative sign when compared to the probability amplitude for both particles being transmitted. This is because the beamsplitter is described by a unitary operator, and one of the reflections carries a negative sign. Classically, this is because if the beamsplitter is built by coating one surface of a dielectric substrate, the reflection coefficient on the air-side (low index) carries a positive sign, but the reflection coeffient on the substrate-side (high index) has a phase flip and picks up a negative sign. Because bosons are symmetric under the exchange of the two particles, the probability amplitudes for the case when both bosons transmit and the case when both bosons reflect have the opposite sign, and for a 50/50 beamsplitter, the two amplitudes add up to zero (destructive interference). As a result, the probability that one boson exits each output port of a beamsplitter is zero. This means that when two identical bosons (such as bosons) interfere art a 50/50 beamsplitter, both photons exit together at one of the output ports, with 50% probability each, and we do not observe a case where one photon exits on each output port of the beamsplitter. This effect is referred to as the Hong-Ou-Mandel interference.

On the other hand, when the two incident particles are fermions, the probability amplitude for both particles being reflected and that for both particles being transmitted carries the same sign. This is because in addition to the negative sign picked up by the case when both particles are reflected at the beam splitter, the exchange of the two particles pick up an additional negative sign, making the two probability amplitude to have the same sign. As a result, the two probability amplitudes interfere constructively, and the two particles *always* exit the beamsplitter one particle on each output. We do not observe a case where both fermionic particles exit the same output port. This phenomenon can be understood based on the Pauli exclusion principle: two identical fermions cannot occupy the same state.

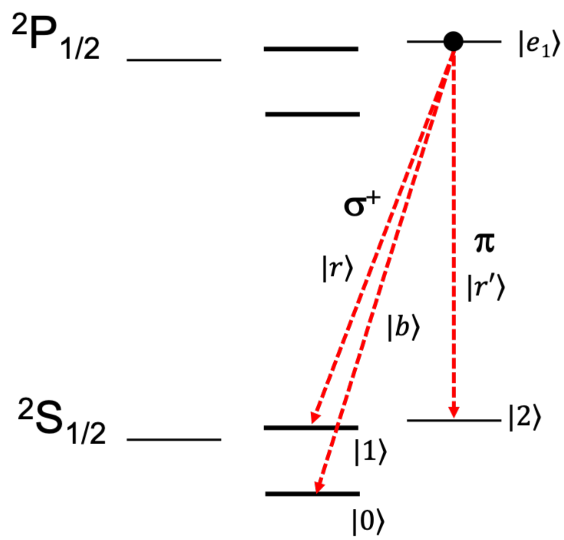

Figure 12.1: An example of atomic level diagram for generating ion-photon entanglement. We consider a hydrogen-like atom with a nuclear spin of $I = 1/2$. The ground state manifold $^2S_{1/2}$ has a hyperfine structure, of which we pick the two $|m_f = 0\rangle$ states to represent the atomic qubit. We also consider the energy levels for the P level, with $J = 1/2$. This level forms a similar hyperfine structure in the excited state manifold $^2P_{1/2}$.

12.2.2 Probabilistic atom-photon entanglement

Let's consider a hydrogen-like atom with one outer electron, and a nucleus with a nuclear spin of $I = 1/2$. The ground state manifold $^2S_{1/2}$ and the first excited state manifold with $J = 1/2$ is shown in Fig. 12.1. Let's consider a scenario where one excites the atom into the $|e_1\rangle = |F = 1, m_F = 1\rangle$ state, shown as the black circle in the figure. This state can spontaneously decay into one of the three states shown by the red dotted lines. In a simple atomic structure like this, the Clebsch-Gordon coefficients dictate that the absolute value of the three coefficients are the same, to be $1/\sqrt{3}$. When it decays into one of the two qubit states ($|0\rangle = |F = 0, m_F = 0\rangle$ or the $|1\rangle = |F = 1, m_F = 0\rangle$ state), the atom emits a photon with σ_+ polarization, with slightly different frequencies corresponding to the hyperfine splitting of the two levels ($|b\rangle$ and $|r\rangle$, respectively). When the atom decays into the $|2\rangle = |F = 1, m_F = 1\rangle$ state, it emits a photon with π polarization ($|r'\rangle$ state).

So, when the atom spontaneously decays from the $|e_1\rangle$ state, the resulting state is given by

$$|\psi_0\rangle = \frac{1}{\sqrt{3}}\left[|0\rangle|b\rangle + |1\rangle|r\rangle + e^{i\varphi}|2\rangle|r'\rangle\right], \tag{12.7}$$

where the first ket denotes the state of the atom and the second the state of the photon characterized by its frequency and polarization (where the unprimed color indicates σ_+ polarization and the primed color indicates π polariztion). φ denotes the phase shift between the $|0\rangle|b\rangle$ and the $|2\rangle|r'\rangle$ states.

The polarization states of the photons σ_+ and π are orthogonal, and practical methods exist that allow us to filter one of the two polarization states. Here, we consider collecting photons with only σ_+ polarization using a polarization filter, and reject photons with π polarization. This is a selection process to reject the probability amplitude of the atom being in the $|2\rangle$ state. Once we apply this polarization filter and consider the cases where the photons with σ_+ polarization are collected, the state can be re-normalized to give

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle|b\rangle + |1\rangle|r\rangle\right]. \tag{12.8}$$

This state indicates maximal entanglement between the atomic qubit state and the state of the emitted photon, indicated by the frequency (or "color") of the photon.

It is important to note that this state is generated *probabilistically*, which is to say that each excitation attempt of the atom does not lead to the generation of this entangled state with unity probability. In practice, we can only collect a small fraction of the photons that are emitted by the atom. Also, we have a finite "protocol efficiency", where any decay events emitting the π polarized photon (and the atom ending up in the $|2\rangle$ state) must be filtered out of the successful cases. This protocol efficiency can be included in the finite collection efficiency of the photon, if the collection optics includes polarization filtering functionality and rejects all the π polarized photons. In a practical experiment, less than 10% of the photons are typically collected, and can be utilized.

12.2.3 Generation of remote atom-atom entanglement

In order to generate entanglement of two atoms between two different locations, both Alice and Bob has to start with an identical experimental setup capable of generating the atom-photon entanglement described in Eq. 12.2.2. We assume that both Alice and Bob can collect a good fraction of the emitted photons into a single mode fiber to clean up the mode, and send the photon through a fiber of length l each, to a location in the middle where the photons in the two fibers interfere at at 50/50 beamsplitter. Alice and Bob synchronizes the excitation events such that the photons arrive at the beamsplitter with perfect temporal overlap to interfere with maximum probability.

Before the beamsplitter, the joint state of Alice and Bob's systems are given by

$$|\psi\rangle_A|\psi\rangle_B = \frac{1}{2}\left[|0\rangle_A|b\rangle_A + |1\rangle_A|r\rangle_A\right]\left[|0\rangle_B|b\rangle_B + |1\rangle_B|r\rangle_B\right], \tag{12.9}$$

where the subscript A (B) denotes atoms or photons from Alice's (Bob's) setup. After the beamsplitter, we place two high efficiency single photon detectors on either output of the beamsplitter, and wait for events where both detectors click. Due to Hong-Ou-Mandel interference, the events corresponding to identical photons arriving at the beamsplitter will not lead to both detectors firing (called the coincidence event). If both detectors fire, it must have come from the cases where the color of the two photons entering the beamplitter were *not* the same. If we only consider the coincident events after the photons are detected, we end up with the resulting atomic state

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}} \left[|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B \right]. \tag{12.10}$$

This signals the generation of an entangled state between Alice's atom and Bob's atom, via entanglement swapping process. A few important notes to keep in mind for this process:

- This entanglement generation process only succeeds with a finite probability (often very small), and is called a probabilistic process.
- The atoms in Alice and Bob's chamber result in entanglement, although the atoms never directly interacted with each other. The entanglement generation is entirely mediated by the photons.
- The successful generation of the entanglement between the atoms is heralded by the coincidence detection, *i.e.*, by the fact that both detectors click. This is critical, as this heralding process automatically filters out all failure cases. When both detectors click, it signals that (1) both photons are emitted by the proper emission process (and not the π polarization photon emission process), (2) that both photons are successfully collected into the fiber, (3) and was not lost during the propagation inside the fiber, (4) successfully interfered at the beamsplitter with the photon of different color from the other side, and (5) was successfully detected by the detector. If any one of these conditions are not fulfilled, the coincidence event is not triggered, and the attempt fails. Alice and Bob will have to prepare their states again and start a new attempt.
- Since we are only trying to generate an entangled pair, no specific quantum information is lost in this attempt.
- Once the entanglement is successfully generated, the entangled pair could be stored in the atomic memory for an extended period of time and can be used to teleport another qubit between the two locations using this entanglement as a resource.
- The teleportation of the data can be in either direction (Alice to Bob, or Bob to Alice). The party intending to send a qubit will perform a Bell state measurement between the qubit to be teleported and her/his qubit in the entangled pair, and send two bits of classical information indicating the measurement result, according to the normal quantum teleportation protocol.