Chapter 1

Angular Momentum in Quantum Mechanics

1.1 Classical vectors vs quantum operators

1.1.1 Relevant definitions and notation

In classical physics, the angular momentum $\boldsymbol{L}$ of a particle is given by the cross product of its position vector $\boldsymbol{r}$ and momentum vector $\boldsymbol{p}$:

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}. \tag{1.1}$$

In quantum physics, we can construct an analogous operator by defining the "vector" operators $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{p}}$ as follows:

$$\begin{aligned} \hat{\boldsymbol{r}} &= \hat{x}_1\boldsymbol{e}_1 + \hat{x}_2\boldsymbol{e}_2 + \hat{x}_3\boldsymbol{e}_3 \\ \hat{\boldsymbol{p}} &= \hat{p}_1\boldsymbol{e}_1 + \hat{p}_2\boldsymbol{e}_2 + \hat{p}_3\boldsymbol{e}_3 \end{aligned} \tag{1.2}$$

where $i = 1, 2, 3$ correspond to x, y, z, resp., and the $\mathbf{e_i}$ are ortho-normal unit vectors in the 3-dimensional vector space. The "hat" ($\hat{}$) symbols signify that these are operators, and the bold type signifies that it is a construction of operators with unit vectors, which we will refer to as vector operators. This notation allows us to perform the classical vector operations, such as dot- and cross-products. In particular, by maintaining the classical equation for the angular momentum vector, we are able to write the $\hat{L}_i$ component of the angular momentum operator given in Eq. 1.1 in the following way:

$$\begin{aligned} \hat{L}_1 &= \hat{x}_2\hat{p}_3 - \hat{x}_3\hat{p}_2 \\ \hat{L}_2 &= \hat{x}_3\hat{p}_1 - \hat{x}_1\hat{p}_3 \\ \hat{L}_3 &= \hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1 \end{aligned} \tag{1.3}$$

and the orbital angular momentum vector operator is $\hat{\boldsymbol{L}} = \hat{L}_1\boldsymbol{e}_1 + \hat{L}_2\boldsymbol{e}_2 + \hat{L}_3\boldsymbol{e}_3$.

Eq. 1.3 can be more compactly written as:

$$\hat{L}_i = \varepsilon_{ijk}\hat{x}_j\hat{p}_k \tag{1.4}$$

where ε_{ijk} is the three-dimensional Levi-Civita symbol, and we employ the Einstein summation convention, i.e. repeated indices imply a sum over that index. This notation will be used in other parts of these notes.

1.1.2 Quantum vector operations

In order to build up a formalism using our quantum vector operators, we need to examine some of their important properties. While the classical position and momentum x_i and p_i commute, this is not the case in quantum mechanics. The commutation relations between position and momentum operators is given by:

$$[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \tag{1.5}$$

where δ_{ij} is the Kronecker delta symbol. It should be noted that the $\boldsymbol{e}_i$ unit vectors commute with all operators.

To see how the vector operators behave under dot and cross products, we can apply them in the normal way:

$$\begin{aligned} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} &= \hat{u}_i \hat{v}_i \\ (\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}})_i &= \varepsilon_{ijk} \hat{u}_j \hat{v}_k \end{aligned} \tag{1.6}$$

Unlike with classical vectors, these are operators, and their order can, in general, not be changed. For the case of the dot-product, switching the order of the vector operators results in:

$$\begin{aligned} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}} &= [\hat{u}_i, \hat{v}_i] + \hat{v}_i \hat{u}_i \\ &= \hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{u}} + [\hat{u}_i, \hat{v}_i] \end{aligned} \tag{1.7}$$

Contrast this with the classical dot product result, where $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$. It is now clear that the classical result holds true for commuting operators, but not in the general case where the operators may not commute. Note the special case of $\hat{\boldsymbol{u}}^2 = \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{u}}$, which does obey the classical dot product relationship.

Similarly, the cross product can we written as:

$$\begin{aligned} (\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}})_i &= \varepsilon_{ijk} \left([\hat{u}_j, \hat{v}_k] + \hat{v}_k \hat{u}_j \right) \\ &= -\varepsilon_{ikj} \hat{v}_k \hat{u}_j + \varepsilon_{ijk} [\hat{u}_j, \hat{v}_k] \\ &= -(\hat{\boldsymbol{v}} \times \hat{\boldsymbol{u}})_i + \varepsilon_{ijk} [\hat{u}_j, \hat{v}_k] \end{aligned} \tag{1.8}$$

which, again, contrasts with the classical result $\boldsymbol{u} \times \boldsymbol{v} = -\boldsymbol{v} \times \boldsymbol{u}$ in the case of non-commuting operators.

If we look at the special cases $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}}$ we see that:

$$(\hat{\boldsymbol{r}} \times \hat{\boldsymbol{r}})_i = (\hat{\boldsymbol{p}} \times \hat{\boldsymbol{p}})_i = \frac{1}{2} \varepsilon_{ijk} \delta_{jk} = 0 \tag{1.9}$$

where the last equality is obvious due to the contradictory requirements of ε_{ijk} and δ_{jk} to be non-zero.

In the case where $\hat{\boldsymbol{u}} = \hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{v}} = \hat{\boldsymbol{p}}$, Eq. 1.7 and Eq. 1.8 result in:

$$\begin{aligned} \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{p}} &= \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{r}} + 3i\hbar \\ \hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}} &= -\hat{\boldsymbol{p}} \times \hat{\boldsymbol{r}} \end{aligned} \tag{1.10}$$

The cross product between $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{p}}$ obeys classical rules, however the dot product does not.

1.1.3 Properties of angular momentum

A simple analysis can be done to show:

$$\begin{aligned} (\hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{v}})^\dagger &= \hat{\boldsymbol{v}}^\dagger \cdot \hat{\boldsymbol{u}}^\dagger \\ (\hat{\boldsymbol{u}} \times \hat{\boldsymbol{v}})^\dagger &= -\hat{\boldsymbol{v}}^\dagger \times \hat{\boldsymbol{u}}^\dagger \end{aligned} \tag{1.11}$$

This property can be used to show that $\hat{\boldsymbol{L}}$ is Hermitian (given that $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{p}}$ are Hermitian):

$$\hat{\boldsymbol{L}}^\dagger = (\hat{\boldsymbol{r}} \times \hat{\boldsymbol{p}})^\dagger = -\hat{\boldsymbol{p}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{L}}. \tag{1.12}$$

It is simple to determine that $\hat{\boldsymbol{L}}$ is orthogonal to both $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{p}}$, just as it is classically:

$$\begin{aligned} \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{L}} &= \hat{x}_i \hat{L}_i, \\ &= \varepsilon_{ijk} \hat{x}_i \hat{x}_j \hat{p}_k = 0, \\ \hat{\boldsymbol{p}} \cdot \hat{\boldsymbol{L}} &= \hat{p}_i \hat{L}_i, \\ &= \varepsilon_{ijk} \hat{p}_i \hat{x}_j \hat{p}_k = 0. \end{aligned} \tag{1.13}$$

The last step of each equation is true due to the sign change of ε_{ijk} under a swap of indices, and the commutation properties of position and momentum operators.

To see how angular momentum commutes with position or momentum, we can use Eq. 1.4 to write:

$$\begin{aligned} [\hat{L}_i, \hat{x}_j] &= \varepsilon_{ikl} \hat{x}_k \hat{p}_l \hat{x}_j - \varepsilon_{ikl} \hat{x}_j \hat{x}_k \hat{p}_l \\ &= \varepsilon_{ikl} \left\{ \hat{x}_k \left(\hat{x}_j \hat{p}_l - i\hbar \delta_{jl} \right) - \hat{x}_j \hat{x}_k \hat{p}_l \right\} \\ &= -i\hbar \varepsilon_{ikj} \hat{x}_k + \varepsilon_{ikl} \left(\hat{x}_k \hat{x}_j \hat{p}_l - \hat{x}_j \hat{x}_k \hat{p}_l \right) \\ &= i\hbar \varepsilon_{ijk} \hat{x}_k \end{aligned} \tag{1.14}$$

The treatment for momentum is very similar, and in summary we obtain the equations:

$$\begin{aligned} [\hat{L}_i, \hat{x}_j] &= i\hbar \varepsilon_{ijk} \hat{x}_k \\ [\hat{L}_i, \hat{p}_j] &= i\hbar \varepsilon_{ijk} \hat{p}_k \end{aligned} \tag{1.15}$$

These equations are said to mean that $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{p}}$ are vectors under rotations.

Using Eq. 1.15 it is possible to derive the commutation relations between different components of $\hat{\boldsymbol{L}}$:

$$\begin{aligned} [\hat{L}_i, \hat{L}_j] &= \hat{L}_i \varepsilon_{jkl} \hat{x}_k \hat{p}_l - \varepsilon_{jkl} \hat{x}_k \hat{p}_l \hat{L}_i \\ &= \varepsilon_{jkl} (\hat{x}_k \hat{L}_i + i\hbar \varepsilon_{iku} \hat{x}_u) \hat{p}_l - \varepsilon_{ikl} \hat{x}_k \hat{p}_l \hat{L}_i \\ &= i\hbar \varepsilon_{jkl} \varepsilon_{iku} \hat{x}_u \hat{p}_l + \varepsilon_{jkl} \hat{x}_k (\hat{p}_l \hat{L}_i + i\hbar \varepsilon_{ilv} \hat{p}_v) - \varepsilon_{ikl} \hat{x}_k \hat{p}_l \hat{L}_i \\ &= i\hbar (\varepsilon_{kjl} \varepsilon_{kiu} \hat{x}_u \hat{p}_l + \varepsilon_{lkj} \varepsilon_{liv} \hat{x}_k \hat{p}_v) \\ &= i\hbar \left[(\delta_{ij} \delta_{lu} - \delta_{il} \delta_{ju}) \hat{x}_u \hat{p}_l + (\delta_{ik} \delta_{jv} - \delta_{ij} \delta_{kv}) \hat{x}_k \hat{p}_v \right] \\ &= i\hbar (\delta_{ij} \hat{x}_l \hat{p}_l - \hat{x}_j \hat{p}_i + \hat{x}_i \hat{p}_j - \delta_{ij} \hat{x}_k \hat{p}_k) \\ &= i\hbar (\hat{x}_i \hat{p}_j - \hat{x}_j \hat{p}_i) \end{aligned} \tag{1.16}$$

where in the fourth step we made use of the identity $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$.

We can write the result in Eq. 1.16 explicitly as:

$$\begin{aligned}
[\hat{L}_x, \hat{L}_y] &= i\hbar\hat{L}_z, \\
[\hat{L}_y, \hat{L}_z] &= i\hbar\hat{L}_x, \\
[\hat{L}_z, \hat{L}_x] &= i\hbar\hat{L}_y,
\end{aligned} \tag{1.17}$$

and which can be compactly written as:

$$[\hat{L}_i, \hat{L}_j] = i\hbar\varepsilon_{ijk}\hat{L}_k. \tag{1.18}$$

It is now clear that different components of the angular momentum do not commute. We can additionally use Eq. 1.18 to straightforwardly show that:

$$[\hat{L}_i, \hat{\boldsymbol{r}}^2] = 0, \quad [\hat{L}_i, \hat{\boldsymbol{p}}^2] = 0, \quad , [\hat{p}_i, \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{p}}] = 0, \quad , [\hat{L}_i, \hat{\boldsymbol{L}}^2] = 0, \tag{1.19}$$

where in particular the last equation (for $\hat{\boldsymbol{L}}^2$) is important. Even though $\hat{L}_i$ does not commute with any other component of $\hat{\boldsymbol{L}}$, it does commute with $\hat{\boldsymbol{L}}^2$.

There is a more elegant way to express the algebra of angular momentum. Using Eq. 1.8 and Eq. 1.18 we can write

$$\begin{aligned}
(\hat{\boldsymbol{L}} \times \hat{\boldsymbol{L}})_i &= -(\hat{\boldsymbol{L}} \times \hat{\boldsymbol{L}})_i + \varepsilon_{ijk}[\hat{L}_j, \hat{L}_k] \\
&= \tfrac{i\hbar}{2}\varepsilon_{jki}\varepsilon_{jkm}\hat{L}_m, \\
&= \tfrac{i\hbar}{2}(\delta_{kk}\delta_{im} - \delta_{km}\delta_{ik})\hat{L}_m, \\
&= \tfrac{i\hbar}{2}(\hat{L}_i - \delta_{ik}\hat{L}_k), \\
&= i\hbar\hat{L}_i,
\end{aligned} \tag{1.20}$$

or, in short

$$\hat{\boldsymbol{L}} \times \hat{\boldsymbol{L}} = i\hbar\hat{\boldsymbol{L}}. \tag{1.21}$$

This result gives an intuitive feel for why angular momentum behaves in an inherently quantum way. A non-zero cross product of a vector with itself is contrary to any classical phenomenom. It is also important to note that, since we only used the general mathematical properties of the cross product of quantum vectors, and the commutation relations of angular momentum operators, Eq. 1.18 and Eq. 1.21 are completely equivalent to each other.

1.2 Angular momentum and central potentials

We now consider the orbital angular momentum in the context of central potentials (potentials that depend only on $r = \sqrt{\hat{\boldsymbol{r}}^2}$), such as the Coulomb potential of an electron in the electric field of an atomic nucleus. The Hamiltonian in this case is given by

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(r), \qquad r = \sqrt{\hat{\boldsymbol{r}}^2}. \tag{1.22}$$

We can write this Hamiltonian in position space using $\hat{\boldsymbol{p}} = -i\hbar\nabla$, or $\hat{\boldsymbol{p}}^2 = -\hbar^2\nabla^2$, where ∇^2 is the Laplace operator. In spherical coordinates, this is

$$\hat{\boldsymbol{p}}^2 = -\hbar^2 \left[\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)\right] \tag{1.23}$$

We can write $\hat{\boldsymbol{L}}^2$ in spherical coordinates as follows:

$$\begin{aligned}
\hat{\boldsymbol{L}}^2 &= \hbar^2(\hat{\boldsymbol{r}} \times \nabla)\cdot(\hat{\boldsymbol{r}} \times \nabla) \\
&= -\hbar^2 \left[\hat{\theta}\left(\tfrac{-1}{\sin\theta}\tfrac{\partial}{\partial\phi}\right) + \hat{\phi}\tfrac{\partial}{\partial\phi}\right]\cdot\left[\hat{\theta}\left(\tfrac{-1}{\sin\theta}\tfrac{\partial}{\partial\phi}\right) + \hat{\phi}\tfrac{\partial}{\partial\phi}\right] \\
&= -\hbar^2 \left[\tfrac{\hat{\theta}}{\sin^2\theta}\left[\hat{\theta}\tfrac{\partial^2}{\partial\phi^2} + \tfrac{\partial\hat{\theta}}{\partial\phi}\tfrac{\partial}{\partial\phi}\right] + \hat{\phi}\tfrac{\partial\hat{\theta}}{\partial\theta}\left(\tfrac{-1}{\sin\theta}\tfrac{\partial}{\partial\phi}\right) + \tfrac{-\hat{\theta}}{\sin\theta}\tfrac{\partial\hat{\phi}}{\partial\phi}\tfrac{\partial}{\partial\theta} + \hat{\phi}\left(\hat{\phi}\tfrac{\partial^2}{\partial\theta^2}\tfrac{\partial\hat{\phi}}{\partial\theta}\tfrac{\partial}{\partial\theta}\right)\right] \\
&= -\hbar^2\left(\tfrac{1}{\sin^2\theta}\tfrac{\partial^2}{\partial\phi^2} + \tfrac{\cos\theta}{\sin\theta}\tfrac{\partial}{\partial\theta} + \tfrac{\partial^2}{\partial\theta^2}\right) \\
&= -\hbar^2\left(\tfrac{1}{\sin\theta}\tfrac{\partial}{\partial\theta}\sin\theta\tfrac{\partial}{\partial\theta} + \tfrac{1}{\sin^2\theta}\tfrac{\partial^2}{\partial\phi^2}\right)
\end{aligned} \tag{1.24}$$

Note that the angular momentum has no r-dependence.

Using Eq. 1.24 and Eq. 1.23, we can rewrite the Hamiltonian as:

$$\hat{H} = \frac{-\hbar^2}{2mr}\frac{\partial^2}{\partial r^2}r + \frac{\hat{\boldsymbol{L}}^2}{2mr^2} + V(r) \tag{1.25}$$

We can now assemble a list of Hermitian operators (observables) we have discussed:

$$\hat{H}, \quad \hat{x}_1, \quad \hat{x}_2, \quad \hat{x}_3, \quad \hat{p}_1, \quad \hat{p}_2, \quad \hat{p}_3, \quad \hat{\boldsymbol{r}}^2, \quad \hat{\boldsymbol{p}}^2, \quad \hat{L}_1, \quad \hat{L}_2, \quad \hat{L}_3, \quad \hat{\boldsymbol{L}}^2 \tag{1.26}$$

where we include all unique Hermitian operators that include up to squared terms of position or momentum.

Typically in atomic physics, the spectrum (energy levels) of an atom is of major interest. We therefore want to only consider observables that commute with the Hamiltonian. This excludes the $\hat{x}_i$ operators (don't commute with $\hat{\boldsymbol{p}}^2$) and $\hat{p}_i$ operators (don't commute with $V(r)$, in general). The same reasoning eliminates $\hat{\boldsymbol{r}}^2$ and $\hat{\boldsymbol{p}}^2$. However, from Eq. 1.25 it is clear that the $\hat{L}_i$ operators and $\hat{\boldsymbol{L}}^2$ commute with H. Due to the $\hat{L}_i$ not commuting with each other, we can only use one of them in our set of operators. The convention for this is $\hat{L}_z$. Our set of commuting Hermitian operators (observables) is then:

$$\hat{H}, \quad \hat{L}_z, \quad \hat{\boldsymbol{L}}^2 \tag{1.27}$$

1.3 Algebra of angular momentum

Hermitian operators $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ are said to satisfy the algebra of angular momentum if they obey the commutation relations given by Eq. 1.18:

$$[\hat{J}_i, \hat{J}_j] = i\hbar\varepsilon_{ijk}\hat{J}_k. \tag{1.28}$$

Keep in mind that this could be orbital angular momentum $\hat{L}_i$, spin angular momentum $\hat{S}_i$, or any other set of Hermitian operators that obeys the above commutation relation. As we showed in section 1.1.3, from this algebra it also follows that:

$$[\hat{J}_i, \hat{\boldsymbol{J}}^2] = 0. \tag{1.29}$$

Define then the following operators:

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y. \tag{1.30}$$

Note that these operators are each other's Hermitian conjugate $\hat{J}_+^\dagger = \hat{J}_-$.

To understand the behavior of these operators and how they act on the angular momentum operators, we can do some simple manipulations:

$$\hat{J}_\pm \hat{J}_\mp = \hat{J}_x^2 + \hat{J}_y^2 \mp i[\hat{J}_x, \hat{J}_y] = \hat{J}_x^2 + \hat{J}_y^2 \pm \hbar \hat{J}_z, \tag{1.31}$$

and so the commutator is

$$[\hat{J}_+, \hat{J}_-] = 2\hbar \hat{J}_z. \tag{1.32}$$

In addition, we can see the following are all equal:

$$\hat{\boldsymbol{J}}^2 - \hat{J}_z^2 \quad = \quad \hat{J}_x^2 + \hat{J}_y^2 \quad = \quad \hat{J}_+\hat{J}_- - \hbar\hat{J}_z \quad = \quad \hat{J}_-\hat{J}_+ + \hbar\hat{J}_z \quad = \quad \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) \tag{1.33}$$

Since we also know that the individual components commute with $\hat{\boldsymbol{J}}^2$ we know that the same must be true for $\hat{J}_\pm$:

$$[\hat{J}_\pm, \hat{\boldsymbol{J}}^2] = 0. \tag{1.34}$$

Lastly, we calculate the commutation with $\hat{J}_z$:

$$\begin{aligned} [\hat{J}_z, \hat{J}_\pm] &= [\hat{J}_z, \hat{J}_x] \pm i[\hat{J}_z, \hat{J}_y] \\ &= i\hbar\hat{J}_y \pm i(-i\hbar)\hat{J}_x \\ &= \pm\hbar\hat{J}_\pm \end{aligned} \tag{1.35}$$

Since each $\hat{J}_i$ and $\hat{\boldsymbol{J}}^2$ commute, we can simultaneously diagonalize them. However, the different $\hat{J}_i$ do not commute with each other, so we have to choose one. Typically, this is $\hat{J}_z$. The simultaneous eigenstates are denoted by $|j, m\rangle$, and they form an orthonormal basis. So far, the only restriction on j and m is $j, m \in \mathbb{R}$.

We label the states in such a way that the following relations apply:

$$\begin{aligned} \hat{\boldsymbol{J}}^2 |j, m\rangle &= \hbar^2 j(j+1) |j, m\rangle \\ \hat{J}_z |j, m\rangle &= \hbar m |j, m\rangle \end{aligned} \tag{1.36}$$

The reason why we choose to define the values using $j(j+1)$ (as opposed to j^2) is for future algebraic convenience, as we will see.

Since it is a requirement that $j(j+1) \geq 0$, it is necessary that either $j \leq -1$ pr $j \geq 0$. However, since the only thing that matters is the eigenvalue, we can just choose $j \geq 0$.

To see how the operators $\hat{J}_\pm$ act on $|j,m\rangle$ we will check the eigenvalue of $\hat{\boldsymbol{J}}^2$ after acting on a state:

$$\hat{\boldsymbol{J}}^2(\hat{J} \pm |j,m\rangle) = \hat{J}_\pm \hat{\boldsymbol{J}}^2 |j,m\rangle = \hbar^2 j(j+1)(\hat{J}_\pm |j,m\rangle). \tag{1.37}$$

From this it is clear that $\hat{J}_\pm$ is still an eigenstate of $\hat{\boldsymbol{J}}^2$ while keeping the eigenvalue j unchanged. However, it is still unknown if it changes the m eigenvalue. To investigate this, we follow the same procedure, but with $\hat{J}_z$:

$$\hat{J}_z(\hat{J}_\pm |j,m\rangle) = (\hat{J}_\pm \hat{J}_z \pm \hbar \hat{J}_\pm) |j,m\rangle = \hbar(m \pm 1)(\hat{J}_\pm |j,m\rangle). \tag{1.38}$$

We see that the m eigenvalue is increased (decreased) by 1 when acted on by $\hat{\boldsymbol{J}}_+$ ($\hat{\boldsymbol{J}}_-$).

Putting together Eq. 1.37 and Eq. 1.38 we can state:

$$\hat{J}_\pm |j,m\rangle = C_\pm(j,m) |j,m \pm 1\rangle \tag{1.39}$$

where $C_\pm(j,m)$ can be determined by taking the inner product of Eq. 1.39 with its Hermitian conjugate:

$$\begin{aligned} \langle j,m\pm 1| C^*_\pm(j,m) C_\pm(j,m) |j,m\pm 1\rangle &= |C_\pm(j,m)|^2, \\ &= \langle j,m| \hat{J}_\mp \hat{J}_\pm |j,m\rangle, \\ &= \langle j,m| \hat{\boldsymbol{J}}^2 - \hat{J}_z^2 \mp \hbar \hat{J}_z |j,m\rangle, \\ &= \hbar^2[j(j+1) - m(m\pm 1)]. \end{aligned} \tag{1.40}$$

This equation shows the convenience of using $j(j+1)$ as the eigenvalue of $\hat{\boldsymbol{J}}^2$. While this equation is simple, we can use it to determine many things about our vector space. For one thing, since $|C_\pm(j,m)|^2$ is the norm of a vector, it must be non-negative, i.e. we must have $j(j+1) - m(m\pm 1) \geq 0$. We can analyze the + and - cases separately.

$$\begin{array}{lllll} (+) & j(j+1) - m(m+1) \geq 0 & \Longrightarrow & -j-1 \leq m \leq j, \\ (-) & j(j+1) - m(m-1) \geq 0 & \Longrightarrow & -j \leq m \leq j+1. \end{array} \tag{1.41}$$

These conditions need to both be true at all times, and so we must use the strictest conditions:

$$-j \leq m \leq j. \tag{1.42}$$

So far there are no requirements on j. However, let's consider a state less than one unit below the maximum value of m, i.e. $|j,m-k\rangle$ for some $0 < k < 1$. Acting on this state with $\hat{J}_+$ gives us a state $|j,m-k+1\rangle$ that is inconsistent with the requirements in Eq. 1.42. The only way to resolve this issue is to ensure that when $\hat{J}_+$ acts on the upper state we obtain the null state, i.e. $C_+(j,m) = 0$. This condition is satisfied when $m = j$. Similarly, $C_-(j,m) = 0$ when $m = -j$. This also means that there need to be an integer number of steps to get from $m = -j$ to $m = j$, and therefore:

$$\begin{gathered} 2j \in \mathbb{N} \quad \rightarrow \quad j \in \mathbb{N}/2 \quad \rightarrow \quad j \in \left\{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \tfrac{5}{2}, ...\right\} \\ m \in \{-j, -j+1, -j+2, ..., j-2, j-1, j\} \end{gathered} \tag{1.43}$$

These are the fundamental quantization properties of angular momentum. They hold for orbital angular momentum, spin angular momentum, and any other space in which Eq. 1.28 is satisfied.