T-S Fuzzy Model with Linear Rule Consequence and PDC Controller: A Universal Framework for Nonlinear Control Systems

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Abstract— **In this paper, we present two results concerning the fuzzy modeling and control of nonlinear systems. The first result is on the approximation of smooth nonlinear dynamical systems using linear Takagi-Sugeno fuzzy models. The second result is on the approximation of smooth nonlinear state-feedback controllers using the the so-called parallel distributed compensation (PDC) controller. Both results are based on the effectiveness of using linear Takagi-Sugeno systems to approximate nonlinear function, which is also proven in this paper.**

I. Introduction

We have witnessed rapidly growing interest in fuzzy control in recent years. This is largely due to the many successful applications of fuzzy control to nonlinear systems. In order to explain the apparent success fuzzy control has enjoyed, it is necessary to investigate the fundamental capabilities of various fuzzy modeling and control frameworks. There has been a great deal of research in addressing this issue [2]-[4], [6]-[7], [16].

Among various fuzzy modeling themes, the so-called Takagi-Sugeno (T-S) model [1] has been one of the most popular modeling framework. A general T-S model employs an affine model with a constant term in the consequent part for each rule. This is often referred as an *affine* T-S model. In this paper, we focus on a special type of T-S fuzzy model in which the consequent part for each rule is represented by a linear model (without a constant term). We refer this type of T-S fuzzy model as a *linear* T-S model. The appeal of a linear T-S model is that it renders itself naturally to Lyapunov based system analysis and design techniques [11] [14]. A commonly held view is that a linear T-S model has limited capability in representing a nonlinear system [8].

In [10], a controller structure called parallel distributed compensation (PDC) is introduced. This structure utilizes a fuzzy state feedback controller which mirrors the structure of the associated linear T-S model. The idea is that for each local linear model, a linear feedback control is designed. The resulting overall controller, which is nonlinear in general, is a fuzzy blending of each individual linear controller. Applications of T-S model together with PDC controller have achieved many successes in real systems [9], [12], [13].

In this paper, we attempt to address the fundamental capabilities of linear T-S models and PDC controllers. Two results are given in this paper. The first result is that a linear Takagi-Sugeno fuzzy model can be an universal approximator of smooth nonlinear dynamic system. It has been known that smooth nonlinear dynamic systems can be approximated by T-S models with affine models as fuzzy rule consequences [3],[6]. However, most results on stability analysis and controller design of T-S models are based on the linear T-S models, i.e., T-S models having linear models as fuzzy rule consequences. The question needed to be addressed is: "Is it possible to approximate any smooth nonlinear systems with Takagi-Sugeno models having linear models as rule consequences?". In paper [5], the authors gave an answer to this question for the simple one-dimensional case. This paper tries to answer this question for n-dimensional nonlinear dynamic system by constructing the T-S model to approximate the original nonlinear system. The answer is "Yes". That is, the original vector field plus its velocity can be accurately approximated if enough number of fuzzy rules are used.

The second result in this paper is that PDC controller can be an universal approximator of nonlinear state-feedback controller. Both results are based on the fact that smooth nonlinear functions under mild constraints can be approximated by linear Takagi-Sugeno systems.

In this paper, $\mathbb{R}^n$ is used to denote the n-dimensional vector spaces of real vectors. C_n^m is used to represent the set of n-dimension functions whose m-th derivative is continuous on the defined region. x_i stands for the i-th component of vector x and $\|\ \|$ stands for the standard vector norm or matrix norm. $O(x)$ is the set of numbers y such that $|\frac{y}{x}| < M$, where M is a constant. $\sum_{j_1 j_2 \ldots j_n}$ is used to represent the summation with all the possible combinations of j_1, $j_2 \ldots$ and j_n.

The paper is organized as follows: Section II gives a construction procedure of T-S system and the proof of the fact that smooth nonlinear function under mild constraints can be accurately approximated by the constructed T-S system. Section III presents the two statements fore-mentioned for T-S model and PDC controller. Concluding remarks are collected in Section IV.

II. Approximation of Nonlinear Functions Using Linear T-S systems

A. Linear T-S Fuzzy Systems

The main feature of linear Takagi-Sugeno fuzzy systems is to express the local properties of each fuzzy implication (rule) by a linear function. The overall fuzzy system is achieved by fuzzy 'blending' of these linear functions. Specifically, the linear Takagi-Sugeno fuzzy system is of the following form:

Rule i: IF $x_1(t)$ is $M_{i1} \cdots$ and $x_n(t)$ is M_{in}
THEN $y = a_i x(t)$,

where $x^T(t) = [x_1(t), x_2(t), \cdots, x_n(t)]$ are the function variables. $i = 1, 2, \cdots, r$ and r is the number of IF-THEN rules. M_{ij} are fuzzy sets. The linear function $y = a_i x(t)$ is the consequence of the i-th IF-THEN rule, where $a_i \in \mathbb{R}^{1\times n}$.

The possibility that the ith rule will fire is given by the product of all the membership functions associated with the ith rule.

$$h_i(x(t)) = \Pi_{j=1}^n M_{ij}(x_j(t)).$$

We will assume that h_i's have already been normalized, i.e. $h_i(x) > 0$ and $\sum_{i=1}^r h_i(x) = 1$. Then by using center of gravity method for defuzzification, we can represent the T-S system as:

$$y = \hat{f}(x) = \sum_{i=1}^r h_i(x) a_i x \qquad (1)$$

The summation process associated with the center of gravity defuzzification in system (1) can also be viewed as an interpolation between the functions $a_i x$ based on the value of the parameter x.

B. Construction Procedure of T-S Fuzzy Systems

Suppose that the nonlinear function $f(x) : \mathbb{R}^n \to \mathbb{R}$ is defined over compact region $D \subset \mathbb{R}^n$ with the following assumptions:

A1-1. $f = 0$

A1-2. $f \in C_1^2$. Therefore, f, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$ are continuous and therefore bounded over D.

Next, we will construct T-S system $\hat{f}(x) = \sum_{i=1}^r h_i(x) a_i x$ to approximate $f(x)$. The objective is to make the approximation error $e(x) = f(x) - \hat{f}(x)$ and its derivative $\frac{\partial e}{\partial x}$ small for all $x \in D$.

Construction Procedures:

1. In region $D_0 = \{x \mid |x_i| < \epsilon_0\}$ where ϵ_0 is a chosen positive number, choose $a_0 = \frac{\partial f}{\partial x}|_{x=0}$.
2. Define the projection operator $P|_x$ mapping $\mathbb{R}^n$ to $n-1$ dimensional subspace $\mathbb{R}^n/x$ as

$$P|_x y = y - \frac{< y, x >}{\|x\|^2} x$$

In region $D \backslash D_0$, choose $x_{j_1 j_2 \ldots . j_n}$ as $[\ j_1\epsilon \quad j_2\epsilon \quad \ldots \quad j_n\epsilon\ ]^T$, where ϵ is a positive number and j_i are integers. Build the linear model $a_{j_1 j_2 \ldots j_n}$ as the solution of the following linear equations:

$$a_{j_1 j_2 \ldots j_n} x_{j_1 j_2 \ldots j_n} = f(x_{j_1 j_2 \ldots . j_n}) \qquad (2)$$

$$a_{j_1 j_2 \ldots j_n} P|_{x_{j_1 j_2 \ldots j_n}} = \frac{\partial f}{\partial x}|_{x_{j_1 j_2 \ldots j_n}} P|_{x_{j_1 j_2 \ldots j_n}} \qquad (3)$$

For fixed $x_{j_1 j_2 \ldots j_n}$, (2)-(3) are n linear equations with the component of $a_{j_1 j_2 \ldots j_n}$ as the variables. (2) implies that f and $\hat{f}$ have the same value at point $x_{j_1 j_2 \ldots j_n}$. (3) implies that $a_{j_1 j_2 \ldots j_n}$ agree with $\frac{\partial f}{\partial x}$ in the $n-1$ dimensional space $\mathbb{R}^n / x_{j_1 j_2 \ldots j_n}$. They are always solvable since x and P are independent of each other, i.e., the matrices $[\ x_{j_1 j_2 \ldots j_n} \quad P|x_{j_1 j_2 \ldots j_n}\ ]$ are always invertible.
3. Choose the fuzzy rules as following:

Rule 0: IF $x_1(t)$ is about $0 \cdots$ and $x_n(t)$ is about 0
THEN $\hat{f}(x) = a_0 x$,

Rule $j_1 j_2 \ldots j_n$: If $x_1(t)$ is about $j_1\epsilon \cdots$ and $x_n(t)$ is about $j_n\epsilon$
THEN $\hat{f}(x) = a_{j_1 j_2 \ldots j_n} x$

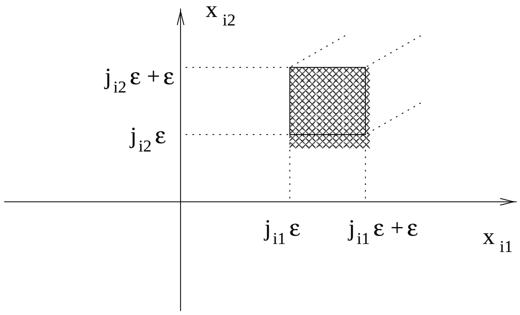

Fig. 1. Projection of $D_{j_1 j_2 \cdots j_n}$ on $x_{i_1} x_{i_2}$ plane

For $Rule$ 0, choose the possibility of firing $h_0(x)$ as 1 inside D_0 and 0 outside. The possibility of firing for the $j_1 j_2 \ldots j_n$th rule is given by the product of all the membership functions associated with the $j_1 j_2 \ldots j_n$-th rule.

$$h_{j_1 j_2 \ldots j_n}(x(t)) = \Pi_{i=1}^{n} M_{j_i}(x_i(t)) \tag{4}$$

where the membership function for x_i is given as

$$M_{j_i}(x_i) = \{ \begin{array}{cc} 1 - \frac{|x_i - j_i \epsilon|}{\epsilon} & |x_i - j_i \epsilon| < \epsilon \\ 0 & \text{else where} \end{array} \tag{5}$$

It is noted that $h_{j_1 j_2 \ldots j_n}(x)$ have already been normalized, i.e. $h_{j_1 j_2 \ldots j_n}(x) \geq 0$ and $\sum_{j_1 j_2 \ldots j_n} h_{j_1 j_2 \ldots j_n}(x) = 1$.

Therefore, we can write $\hat{f}(x)$ as:

$$\hat{f}(x) = h_0 a_0 x + \sum_{j_1 j_2 \ldots j_n} h_{j_1 j_2 \ldots j_n} a_{j_1 j_2 \ldots j_n} x \tag{6}$$

Remark: It should be pointed out that the specific membership function constructed above is only needed when we want to approximate both the nonlinear function and its derivative. There will be much more freedom if we only need to approximate the function itself.

C. Analysis of Approximation

In this subsection, we will prove the fact that any smooth nonlinear function can be approximated, to any degree of accuracy, using the linear T-S fuzzy systems constructed above. This fact will form the foundation of the two statements in this paper.

First, we will divide region $D \backslash D_0$ into many small regions

$$D_{j_1 j_2 \ldots j_n} = \{x |\, x \in D,\, j_i \epsilon \leq x_i \leq (j_i + 1)\epsilon \ \ \forall i\}$$

In the following discussions, we will only concentrate on one of such regions ($D_{j_1 j_2 \ldots j_n}$), which is shown in Fig. 1, by assuming that $x \in D_{j_1 j_2 \ldots j_n}$. From the construction procedure above, we know that only the fuzzy rules centered at the vertices of $D_{j_1 j_2 \ldots j_n}$ can be activated at x. That is $h_{l_1 l_2 \ldots l_n}(x) \neq 0$ only if $x_{l_1 l_2 \ldots l_n}$ is one of the vertex points of $D_{j_1 j_2 \ldots j_n}$.

Consider $e(x)$, the approximation error between $f(x)$ and $\hat{f}(x)$

$$\begin{aligned} \|e(x)\| &= \|f(x) - \sum_{j_1 j_2 \ldots j_n} h_{j_1 j_2 \ldots j_n}(x) a_{j_1 j_2 \ldots j_n} x\| \\ &= \|f(x) - \sum_{j_1 j_2 \ldots j_n} h_{j_1 j_2 \ldots j_n}(x) a_{j_1 j_2 \ldots j_n} x_{j_1 j_2 \ldots j_n} \\ &\quad - \sum_{j_1 j_2 \ldots j_n} h_{j_1 j_2 \ldots j_n}(x) a_{j_1 j_2 \ldots j_n}(x - x_{j_1 j_2 \ldots j_n})\| \\ &= \|f(x) - \sum_{j_1 j_2 \ldots j_n} h_{j_1 j_2 \ldots j_n}(x) f(x_{j_1 j_2 \ldots j_n}) - \\ &\quad \sum_{j_1 j_2 \ldots j_n} h_{j_1 j_2 \ldots j_n}(x) a_{j_1 j_2 \ldots j_n}(x - x_{j_1 j_2 \ldots j_n})\| \\ &\leq \sum_{j_1 j_2 \ldots j_n} h_{j_1 j_2 \ldots j_n}(x) \|f(x) - f(x_{j_1 j_2 \ldots j_n})\| + \\ &\quad \sum_{j_1 j_2 \ldots j_n} h_{j_1 j_2 \ldots j_n}(x) \|a_{j_1 j_2 \ldots j_n}(x - x_{j_1 j_2 \ldots j_n})\| \\ &\leq \max_{l_1 l_2 \ldots l_n} \|f(x) - f(x_{l_1 l_2 \ldots l_n})\| + \\ &\quad \max_{l_1 l_2 \ldots l_n} \|a_{l_1 l_2 \ldots l_n}(x - x_{l_1 l_2 \ldots l_n})\| \end{aligned}$$

Note that

$$\begin{aligned} & a_{l_1 l_2 \ldots l_n}(x - x_{l_1 l_2 \ldots l_n}) \\ = \ & \frac{\langle (x - x_{l_1 l_2 \ldots l_n}), x_{l_1 l_2 \ldots l_n} \rangle}{\|x_{l_1 l_2 \ldots l_n}\|^2} f(x_{l_1 l_2 \ldots l_n}) + \frac{\partial f}{\partial x} |_{x_{l_1 l_2 \ldots l_n}} \\ & \left((x - x_{l_1 l_2 \ldots l_n}) - \frac{\langle (x - x_{l_1 l_2 \ldots l_n}), x_{l_1 l_2 \ldots l_n} \rangle}{\|x_{l_1 l_2 \ldots l_n}\|^2} x_{l_1 l_2 \ldots l_n} \right) \end{aligned}$$

Since $x \in D_{j_1 j_2 \ldots j_n}$, the distance between x and any vertex point of $D_{j_1 j_2 \ldots j_n}$ is less than $\sqrt{n}\epsilon$, i.e. $| \ x - x_{l_1 l_2 \ldots l_n} \ | \leq \sqrt{n}\epsilon$, we can make $e(x)$ arbitrarily small by just reducing ϵ.

Now consider the approximation of $\frac{\partial f}{\partial x}$. Before doing that, three facts for the membership functions are presented.

Fact 1: Define

$$\frac{\partial h_{j_1 j_2 \ldots j_n}}{\partial x} |_x = \left[\begin{array}{cccc} \frac{\partial h_{j_1 j_2 \cdots j_n}}{\partial x_1} |_x & \frac{\partial h_{j_1 j_2 \cdots j_n}}{\partial x_2} |_x & \ldots & \frac{\partial h_{j_1 j_2 \cdots j_n}}{\partial x_n} |_x \end{array} \right]$$

where it exists, then

$$\sum_{j_1 j_2 \ldots j_n} \frac{\partial h_{j_1 j_2 \ldots j_n}}{\partial x} |_x = 0 \tag{7}$$

Proof: Take the derivatives of $\sum_{j_1j_2...j_n} h_{j_1j_2...j_n}$. Since $\sum_{j_1j_2...j_n} h_{j_1j_2...j_n} = 1$, its derivatives with respect to x_i will be 0. ■

Fact 2:

$$\sum_{j_1j_2...j_n} (x - x_{j_1j_2...j_n}) \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x = -I \quad (8)$$

Proof: For vertex point $x_{l_1l_2...l_n} \in D_{j_1j_2...j_n}$, define $\bar{l}_i = 2j_i + 1 - l_i$, then it can be proven that

$$(x - x_{l_1l_2...\bar{l}_i...l_n})_i \frac{\partial h_{l_1l_2...\bar{l}_i...l_n}}{\partial x_i} |_x + (x - x_{l_1l_2...l_n})_i \frac{\partial h_{l_1l_2...l_n}}{\partial x_i} |_x = -(h_{l_1l_2...l_n} + h_{l_1l_2...\bar{l}_i...l_n})$$

$$(x - x_{l_1l_2...\bar{l}_i...l_n})_i \frac{\partial h_{l_1l_2...\bar{l}_i...l_n}}{\partial x_j} |_x + (x - x_{l_1l_2...l_n})_i \frac{\partial h_{l_1l_2...l_n}}{\partial x_j} |_x = 0, \quad i \neq j$$

Summing up these equations for all the rules $l_1l_2...l_n$ that are effective in region $D_{j_1j_2...j_n}$, the fact is proved. ■

Fact 3: Define a_x as the solution of the following linear equations

$$a_x x = f(x) \quad (9)$$

$$a_x P |_x = \frac{\partial f}{\partial x} |_x P \quad (10)$$

Then $\forall \delta$, $\exists \epsilon$ such that $\|a_x - a_{j_1j_2...j_n}\| \leq \delta$ if $\|x - x_{j_1j_2...j_n}\| \leq \epsilon \ll 1$,

Proof: Since a_x is the solution of the linear equations (9) (10) and all the parameters of the equations ($f(x)$, $\frac{\partial f}{\partial x}$ and $P|_x$) are continuous functions of x, a_x will depend continuously on x. Consequently, $\|a - a_{j_1j_2...j_n}\|$ can be made arbitrarily small by choosing a small enough value for ϵ. ■

Now consider $\frac{\partial e}{\partial x}$, the difference between $\frac{\partial f}{\partial x}$ and $\frac{\partial \hat{f}}{\partial x}$.

$$\begin{aligned}
\|\frac{\partial e}{\partial x}\| &= \|\frac{\partial f}{\partial x} |_x - \frac{\partial (\sum_{j_1j_2...j_n} h_{j_1j_2...j_n} a_{j_1j_2...j_n} x)}{\partial x}\| \\
&= \|\frac{\partial f}{\partial x} |_x - \sum_{j_1j_2...j_n} a_{j_1j_2...j_n} x \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x \\
&\quad - \sum_{j_1j_2...j_n} h_{j_1j_2...j_n}(x) a_{j_1j_2...j_n}\| \\
&= \|\frac{\partial f}{\partial x} |_x - \sum_{j_1j_2...j_n} f(x_{j_1j_2...j_n}) \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x \\
&\quad - \sum_{j_1j_2...j_n} a_{j_1j_2...j_n} (x - x_{j_1j_2...j_n}) \frac{\partial h_{j_1j_2...j_n}}{\partial x} \\
&\quad - \sum_{j_1j_2...j_n} h_{j_1j_2...j_n}(x) a_{j_1j_2...j_n} |_x \| \\
&= \|\frac{\partial f}{\partial x} |_x - \sum_{j_1j_2...j_n} (f(x) + \frac{\partial f}{\partial x} |_x (x_{j_1j_2...j_n} - x) + \\
&\quad O(\epsilon^2)) \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x - \sum_{j_1j_2...j_n} h_{j_1j_2...j_n}(x) a_{j_1j_2...j_n} \\
&\quad - \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x \sum_{j_1j_2...j_n} a_{j_1j_2...j_n} (x - x_{j_1j_2...j_n})\| \\
&= \|\frac{\partial f}{\partial x} |_x - \sum_{j_1j_2...j_n} \frac{\partial f}{\partial x} |_x (x_{j_1j_2...j_n} - x) \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x \\
&\quad - \sum_{j_1j_2...j_n} a_{j_1j_2...j_n} (x - x_{j_1j_2...j_n}) \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x - \\
&\quad \sum_{j_1j_2...j_n} h_{j_1j_2...j_n}(x) a_{j_1j_2...j_n}\| + O(\epsilon) \\
&\quad \text{(From Fact 1)} \\
&= \| - \sum_{j_1j_2...j_n} a_{j_1j_2...j_n} (x - x_{j_1j_2...j_n}) \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x - \\
&\quad \sum_{j_1j_2...j_n} h_{j_1j_2...j_n}(x) a_{j_1j_2...j_n}\| + O(\epsilon) \\
&\quad \text{(From Fact 2)} \\
&= \| \sum_{j_1j_2...j_n} a_{j_1j_2...j_n} (x - x_{j_1j_2...j_n}) \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x + a_x \\
&\quad + \sum_{j_1j_2...j_n} h_{j_1j_2...j_n} a_{j_1j_2...j_n} - \sum_{j_1j_2...j_n} h_{j_1j_2...j_n}(x) a_x\| \\
&\quad + O(\epsilon) \\
&\leq \| \sum_{j_1j_2...j_n} (a_{j_1j_2...j_n} - a_x)(x - x_{j_1j_2...j_n}) \frac{\partial h_{j_1j_2...j_n}}{\partial x} |_x \| \\
&\quad + \| \sum_{j_1j_2...j_n} h_{j_1j_2...j_n} (a_{j_1j_2...j_n} - a_x)\| + O(\epsilon) \\
&\quad \text{(From Fact 2)}
\end{aligned}$$

From Fact (3), it is known $\frac{\partial e}{\partial x}$ can be made arbitrarily small by reducing ϵ.

Next consider region D_0. In region D_0, it is known from Taylor series that $e(x)$ and $\frac{\partial e}{\partial x}$ can also be made arbitrarily small by reducing ϵ_0. Therefore, we have the following theorem by summarizing the results above:

Theorem 4: For smooth nonlinear function $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^1$ satisfied Assumptions A1-1 A1-2, it can be approximated, to any degree of accuracy, by linear T-S fuzzy systems. Furthermore, its derivatives can be approximated to any degree of accuracy, except for a finite number of points.

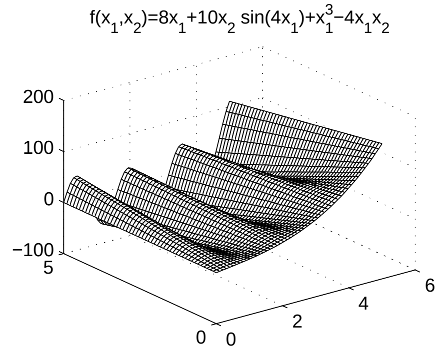

Fig. 2. Nonlinear function $f(x_1, x_2) = 8x_1 + 10x_2 \sin(4x_1) + x_1^3 - 4x_1x_2$

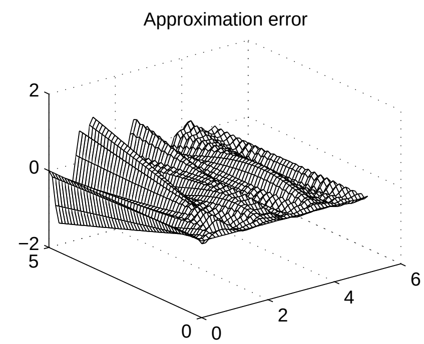

Fig. 3. Approximation error of nonlinear function

Remark: It may be argued that the membership function is not continuous on the boundary between D_0 and $D_{j_1 j_2 \ldots j_n}$. To overcome the discontinuity, some bumper functions can be included to smooth the membership function without affecting the approximation accuracy [15].

D. Example

An example is given in this subsection for illustration. Consider the approximation of two dimensional nonlinear function $f(x_1, x_2) = 8x_1 + 10x_2 \sin(4x_1) + x_1^3 - 4x_1x_2$ shown in Figure 2. A 25×40 grid is used. The maximum approximation error is 1.38. We also plot the approximation error in Figure 3. It should be pointed out the approximation error could be further reduced by using more fuzzy rule.

III. Applications to Modeling and Control of Nonlinear Systems

A. Approximation of Nonlinear Dynamic Systems using Linear Takagi-Sugeno Fuzzy Models

The linear Takagi-Sugeno fuzzy model is used to describe dynamic systems. It is of the following form:

Rule i: IF $x_1(t)$ is M_{i1}, $\cdots$ and $x_n(t)$ is M_{in}
THEN $\dot{x}(t) = A_i x(t)$,

where $x^T(t) = [x_1(t), x_2(t), \cdots, x_n(t)]$ are the system states. $i = 1, 2, \cdots, r$ and r is the number of IF-THEN rules. M_{ij} are fuzzy sets and $\dot{x}(t) = A_i x(t)$ are the consequences of the i-th IF-THEN rule.

By using center of gravity method for defuzzification, we can represent the T-S model as:

$$\dot{x} = \hat{f}(x) = \sum_{i=1}^{r} h_i(x) A_i x \tag{11}$$

where $h_i(x)$ is the possibility for the i-th rule to fire.

Consider the nonlinear system:

$$\dot{x} = f(x) \tag{12}$$

where $f(x)$ is a vector field defined over compact region $D \subset \mathbb{R}^n$ with the following assumptions:

A2-1 $f(0) = 0$, i.e. the origin is an equilibrium point.

A2-2 $f \in C_n^2$. Therefore, f, $\frac{\partial f}{\partial x}$, and $\frac{\partial^2 f}{\partial x^2}$ are continuous and bounded over D.

Suppose $f(x)$ can be written as $\begin{bmatrix} f_1(x) & \ldots & f_n(x) \end{bmatrix}^T$. What we mean by approximation is finding a T-S fuzzy model $\hat{f}(x) = \begin{bmatrix} \hat{f}_1(x) & \ldots & \hat{f}_n(x) \end{bmatrix}^T$ such that $\|f(x) - \hat{f}(x)\|$ is small. Since $\|f(x) - \hat{f}(x)\|$ is small if and only if each of its components (which are nonlinear functions) is small, then by applying Theorem 4 proven in the previous section, we obtain the following corollary:

Corollary 5: For any smooth nonlinear system (12) satisfying the above assumptions, it can be approximated, to any degree of accuracy, by a T-S model (11).

Similarly, smooth nonlinear control system $\dot{x} = f(x) + g(x)u$ can also be approximated using a T-S fuzzy model $\dot{x} = \sum_{i=1}^{r} h_i(x)(A_i x + B_i u)$. By treating u as extraneous system state, we can also approximate the smooth nonlinear control system $\dot{x} = f(x, u)$ by T-S fuzzy model $\dot{x} = \sum_{i=1}^{r} \hat{h}_i(x, u)(A_i x + B_i u)$. In this case, the fuzzy rule is of the following form:

Rule i: IF $x_1(t)$ is M_{i1}, $\cdots$, $x_n(t)$ is M_{in}, $u_1(t)$ is N_{i1}, $\cdots$ and $u_m(t)$ is N_{im}
THEN $\dot{x}(t) = A_i x(t) + B_i u(t)$,

where $x^T(t) = [x_1(t), x_2(t), \cdots, x_n(t)]$ are the system states and $u^T(t) = [u_1(t), u_2(t), \cdots, u_m(t)]$ are the system inputs. $i = 1, 2, \cdots, r$ and r is the number of

IF-THEN rules. M_{ij}, N_{ij} are fuzzy sets and $\dot{x}(t) = A_i x(t) + B_i u(t)$ is the consequence of the i-th IF-THEN rule. $\hat{h}_i(x,u) = \Pi_{j=1}^{n} M_{ij}(x_i(t)) \Pi_{k=1}^{m} N_{ik}(u_k(t))$ is the possibility for the i-th rule to fire.

Remark: There are many results on the approximation of a nonlinear system using a T-S model with affine models as rule consequences. Instead of using linear models $\dot{x} = A_i x(t) + B_i u(t)$ as rule consequences, affine models $\dot{x} = A_i x(t) + B_i u(t) + C_i$ are used. To do that, the state space is first divided into many small regions and first-order Taylor expansion around a point in that region is adopted as the rule consequence. By including the constant term in the output of the fuzzy rules, more flexibility can be obtained in the choice of regions and membership functions, but the stability analysis and synthesis become more involved.

B. Approximation of Nonlinear State-Feedback Controller using PDC Controller

In this paper, we will consider a special form of fuzzy controller introduced in [11] where it was termed parallel distributed compensation (PDC). The PDC controller structure consists of the fuzzy rules:

Rule j: IF $x_1(t)$ is $M_{j1} \cdots$ and $x_n(t)$ is M_{jn}
THEN $u(t) = K_j x(t)$

where $j = 1, 2, \cdots, s$. The output of the PDC controller is

$$u = \sum_{j=1}^{s} h_j(x) K_j x \tag{13}$$

Following similar argument as in the above subsection, we obtain the following theorem:

Theorem 6: For smooth nonlinear state feedback controller $u = K(x)$ defined over a compact region ($K(0) = 0$), can be approximated to any degree of accuracy by PDC controller (13).

IV. Conclusions

In this paper, we discussed the approximation of nonlinear system using T-S models with linear models as rule consequences. We presented a construction procedures of T-S models and proved that this constructed T-S model can approximate smooth nonlinear system plus its velocity with any desired accuracy. We also showed that the effectiveness of PDC controller as an universal approximator of smooth nonlinear state-feedback controller.

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