

**Stability Analysis of 1st and 2nd
Order Sigma Delta Analog
to Digital Converter**

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OVERVIEW:

Sigma Delta Analog-to-Digital converter(ADC) utilizes over-sampling and digital filtering to produce a high resolution digitized output. In this two step process, an analog sigma delta modulator (SDM) first samples the input signal and produces a single bit pulse coded modulated output. Next, a digital low pass filter removes the high frequency quantization noise introduced by the modulator, decimates the data sequence, and gives out N bit digital output. N is resolution of ADC. The resulting output of ADC is a highly accurate digital representation of the analog input signal.

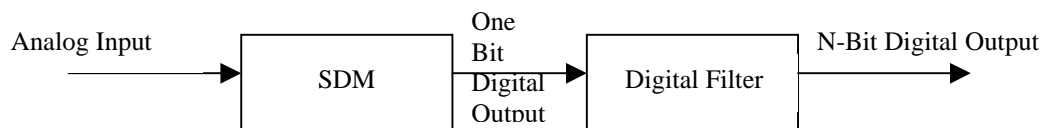


Figure 1: *Analog to Digital Converter*

NATURE OF PROBLEM:

SDM consists of a 1-bit quantizer, which is non-linear element. To properly model and analyze the stability of Sigma-Delta modulator several non-linear techniques are used, such as:

1. Using describing function to analyze the stability of non-linear feedback systems.
2. Representing the ADC with a set of Ordinary Differential Equations. We are using 2nd approach to address the stability.

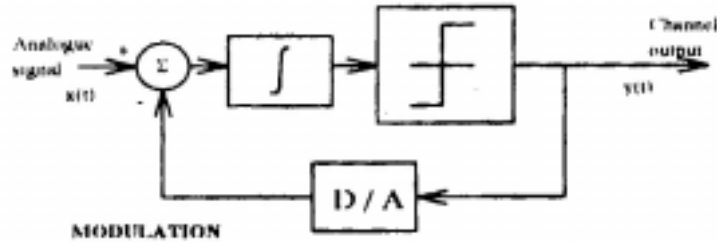


Figure 2: *1st-Order Sigma Delta Converter*

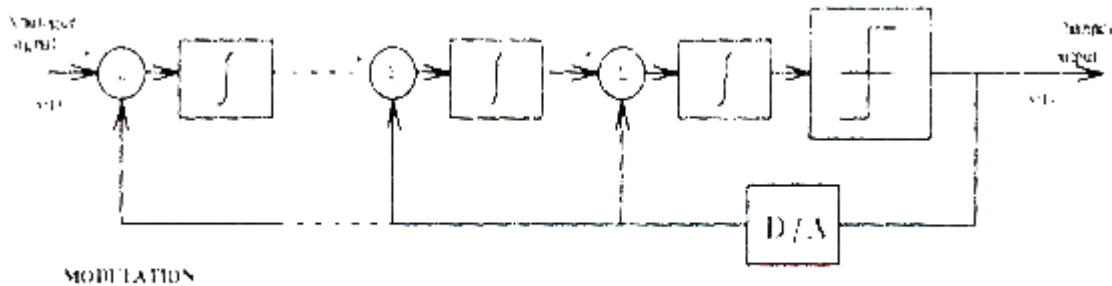


Figure 3: *Nth-Order Sigma Delta Converter*

We are concerned with the stability of one bit output of SDM and not the N-Bit output of ADC.

CONCEPT OF STABILITY FOR SIGMA DELTA:

The traditional stability definitions such as Bounded Input Bounded Output (BIBO) are not valid for the ADC as the clipping due to quantizer will always keep output voltage bounded. “An accurate and simple to use formula for the stable input limit would be of immense value in the design of delta-sigma modulators. Unfortunately, such a formula is currently unavailable.” [4] As per our understanding, stability analysis of SDM can be done w.r.t. time varying signals

such as sinusoidal input or time constant signals such as DC input. SDM, which is stable for one-volt DC input, can exhibit instability for a one-volt sinusoidal input at 1 kHz. Similarly SDM which is stable for a one-volt sinusoidal input at 1 kHz could become unstable at 0.2 volts at the same frequency. This makes the stability analysis of SDM quite complex. From the practical point of view the output of an unstable sigma delta AD converter consists of repeated cycles of ones or zeros or a combination of ones and zeros for a long period of time.

ANALYSIS DOMAIN:

The stability analysis can be done in either discrete (time or frequency) domain or continuous (time or frequency) domain. We have chosen stability analysis in the continuous time domain. Hence, we have to model the sampled SDM as a continuous time model. This necessitates the modeling of the quantizer as a continuous time element. We have chosen the Tanh function to model the quantizer in this manner as shown in Figure 4.

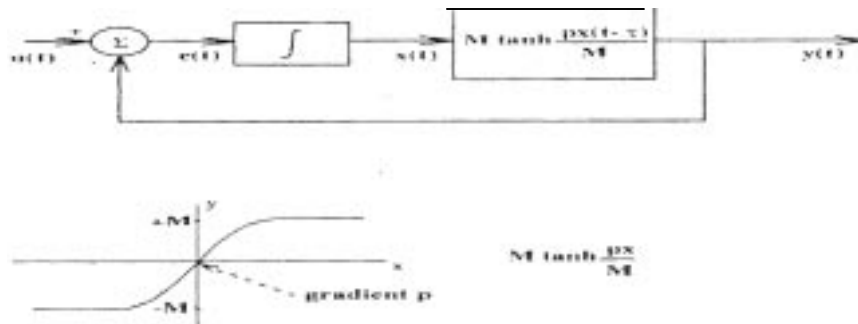


Figure 4: *Continuous Time Model of One-Bit SDM*

Courses such as ‘Random Signals and Noise’ and ‘Digital Signal Processing’ model the quantizer as a noise source, which is uncorrelated with the input signal, as shown in Figure 5.

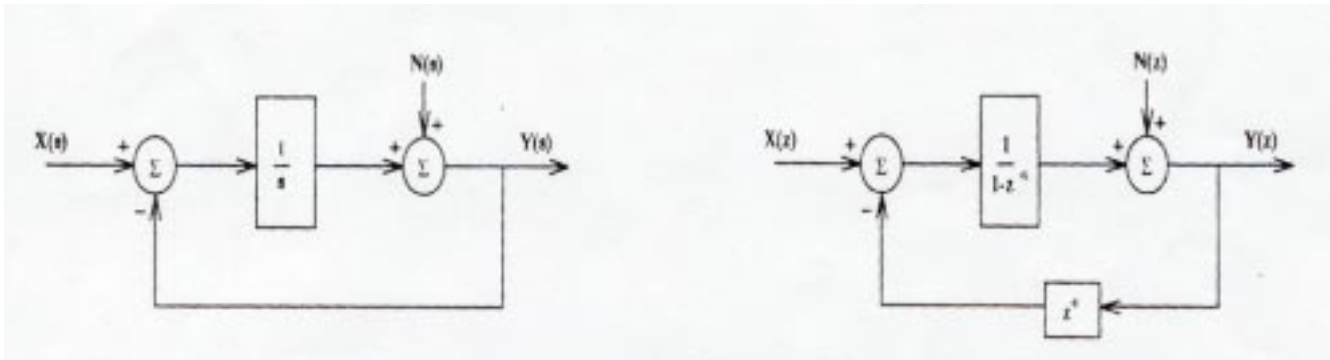


Figure 5: *Continuous and Discrete Frequency Models of One-Bit SDM*

ANALYSIS NODE:

From the numerical analysis point of view, a repetitive combination of zeros and ones at the output of the SDM doesn't convey whether the system is stable or unstable. Hence the stability analysis can not be done at the output node. Instead it must be done at the input to the quantizer or in other words the output of the integrator.

APPROACH:

The Sampled Second Order Sigma Delta modulator is represented with an equivalent "Continuous Second Order Sigma Delta Modulator" and a set of

Ordinary Differential Equations are used to represent the continuous system. [1]. The 1 bit quantizer is represented as a hyperbolic tangent function with a sufficiently steep gradient in the crossover region. [2] This paper is concerned with an analysis of these equations particularly w.r.t. gain of the feedback loop and the integrator leakage. 4th Order Runge Kutta iterative process is used to solve the ODEs representing the second order Continuous Sigma Delta Modulator.

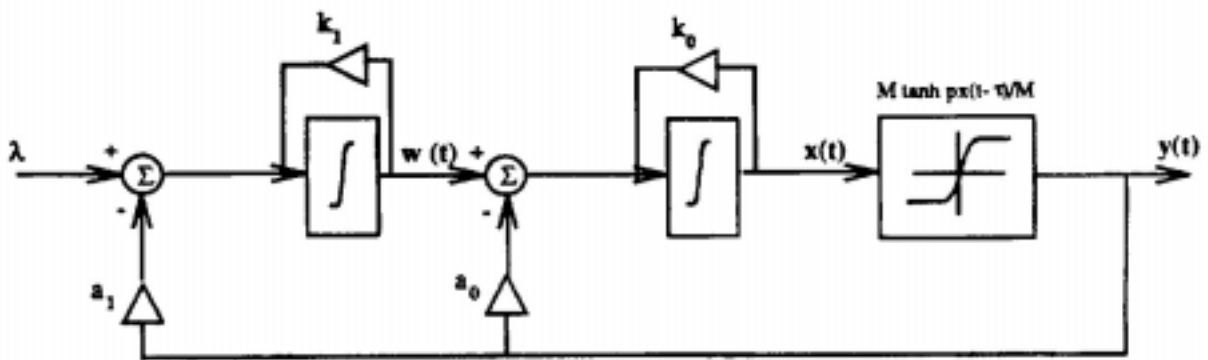


Figure 6: *Continuous Time Model of Second Order SDM*

a_0 and a_1 are the gains of the second and first loops, respectively.
 k_0 and k_1 are the integrator leakage coefficients for their respective integrators.

The parameters for the continuous time model of second order SDM:

1. a_0, a_1
2. k_0, k_1
3. M (bound level of quantizer)
4. p (slope of Tanh function at 0)
5. amp (amplitude of input Sin wave)
6. τ (time step of input sampling)
7. f (frequency of input Sin wave)
8. dc (dc level of input signal)

DERIVATION OF SYSTEM EQUATIONS:

at comparator :

$$e(t) = u(t) - y(t) \quad (2)$$

where :

$$y(t) = M \tanh \frac{px(t - \tau)}{M}$$

so equating (1) and (2):

$$e(t) = u(t) - M \tanh \frac{px(t - \tau)}{M} = \frac{dx(t)}{dt} + kx(t)$$

giving first order modulation ODE :

$$\frac{dx(t)}{dt} = u(t) - kx(t) - M \tanh \frac{px(t - \tau)}{M}$$

Similarly equations can be formulated for higher order modulators.

Second order modulation ODEs :

$$\frac{dx(t)}{dt} = w(t) - kx(t) - M \tanh \frac{px(t - \tau)}{M}$$

$$\frac{dw(t)}{dt} = u(t) - kw(t) - M \tanh \frac{px(t - \tau)}{M}$$

Third order modulation ODEs :

$$\frac{dx(t)}{dt} = w(t) - kx(t) - M \tanh \frac{px(t - \tau)}{M}$$

$$\frac{dw(t)}{dt} = v(t) - kw(t) - M \tanh \frac{px(t - \tau)}{M}$$

$$\frac{dv(t)}{dt} = u(t) - kv(t) - M \tanh \frac{px(t - \tau)}{M}$$

$$\frac{dx(t)}{dt} = w(t) - k_0x(t) - a_0M \tanh \frac{px(t - \tau)}{M} \quad (1)$$

$$\frac{dw(t)}{dt} = \lambda - k_1w(t) - a_1M \tanh \frac{px(t - \tau)}{M} \quad (2)$$

System of Ordinary Differential Equations representing a continuous second order SDM

CONCLUSIONS:

When p is small then $\text{Tanh}(px)$ is approx. equal to px . Hence, the output of the quantizer is a scaled version of the output of the integrator. A p value of $2e7$ was found to be appropriate for our testing. We are unsure of the value of p chosen by the authors of references [1,2,3]. The results of simulation of 1st order and 2nd order SDM along with the Matlab code are attached in Appendix I.

REFERENCES:

- 1) G. Ushaw, S. McLaughlin, "On the configuration of second order sigma delta modulator", IEE Colloquium on Oversampling Techniques and Sigma-Delta Modulation',1994.
- 2) G. Ushaw, S. McLaughlin, "On the stability and configuration of sigma delta modulators", 1994 IEEE International Symposium on Circuits and Systems, vol. 5, pp. 349-352, 1994.
- 3) G. Ushaw, S. McLaughlin, Hughes D. T., Mulgrew B. "A representation of sigma delta modulators as continuous systems for analysis of their effect on chaotic signals", Proceedings of the SPIE, vol. 2038, pp. 103-14, July 1993.
- 4) R. Schreier, "An Empirical Study of High-Order Single-Bit Delta-Sigma Modulators", IEEE Transactions on Circuits and Systems- II: Analog and Digital Signal Processing, Vol. 40, No. 8, pp. 461-466, August 1993.

APPENDIX I

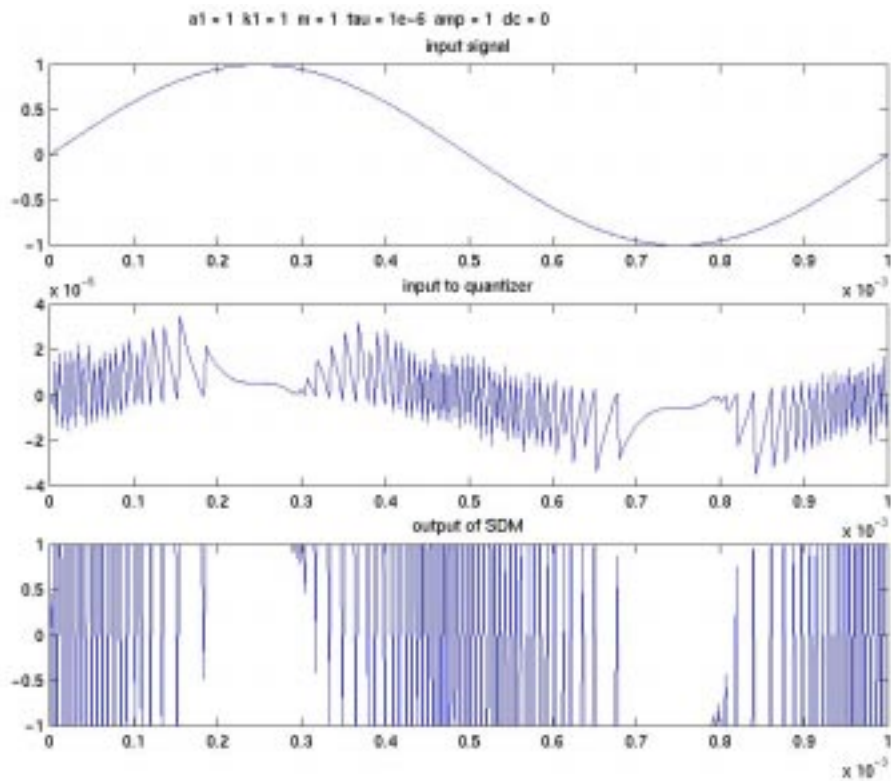


Figure1: The input to quantizer is of lower frequency when the slope of input approaches zero. Since, the input to quantizer is bounded, the system is stable.

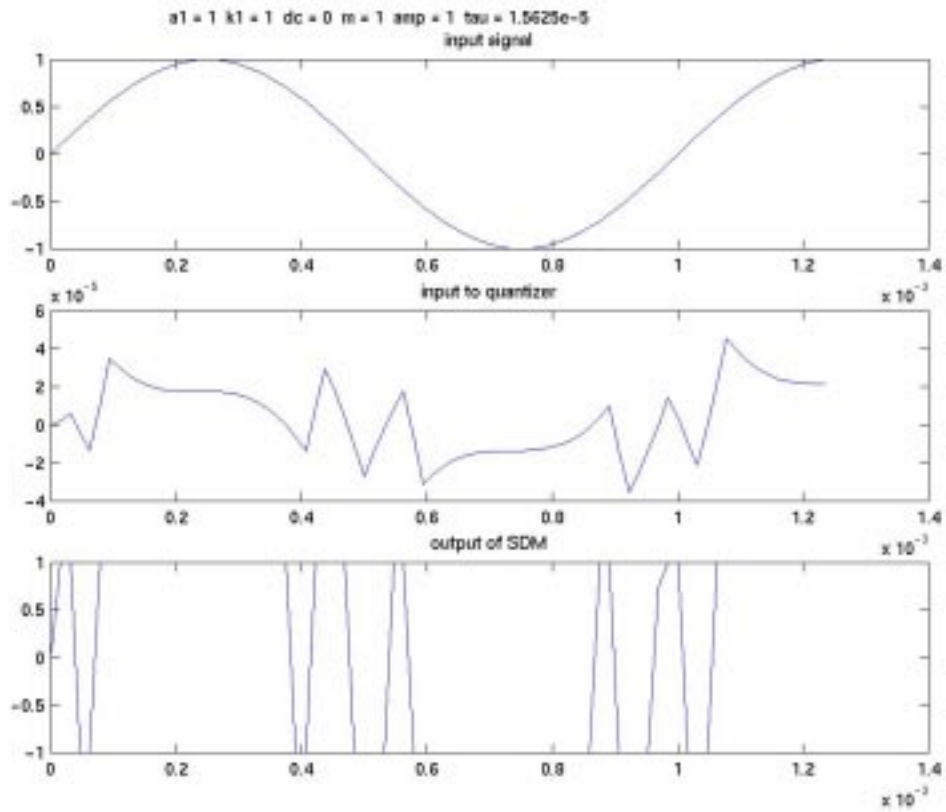


Figure 2: Sampling frequency reduced to 64 times that of signal Frequency. It can be seen that the accuracy of output of SDM is reduced as compared to Figure 1. The system is still stable as the input to quantizer is bounded.

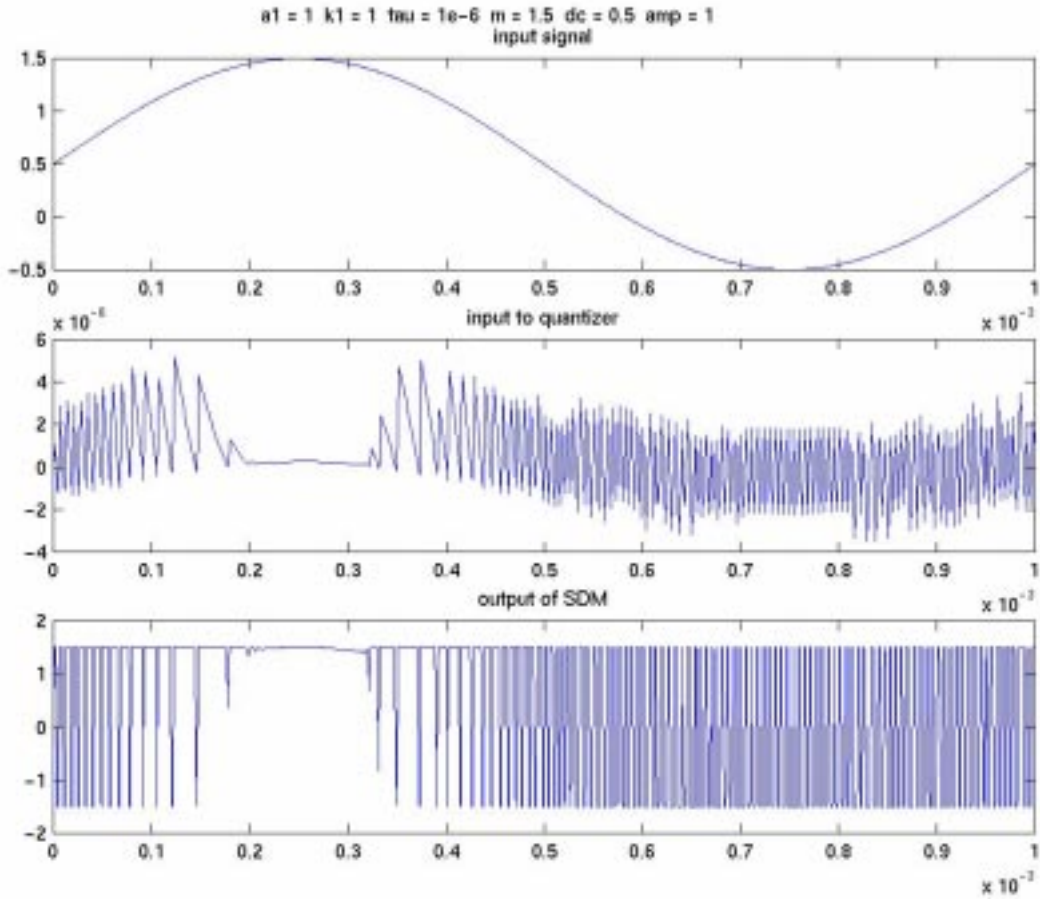


Figure 3: Sinusoid superimposed on dc signal. M has been adjusted to keep the quantization level correct. The system is stable.

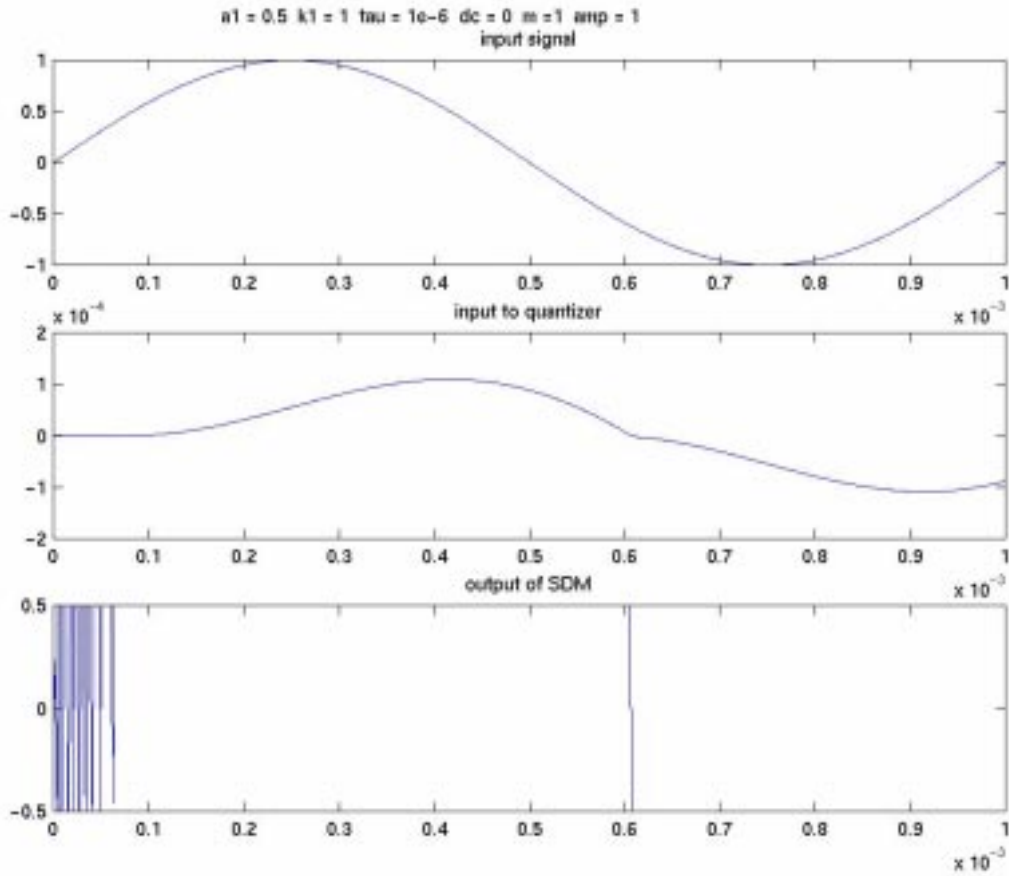


Figure 4: Gain of the feedback loop (a_1) reduced to 0.5. System is still stable but the accuracy of the output of SDM is drastically reduced as compared to when $a_1=1$.

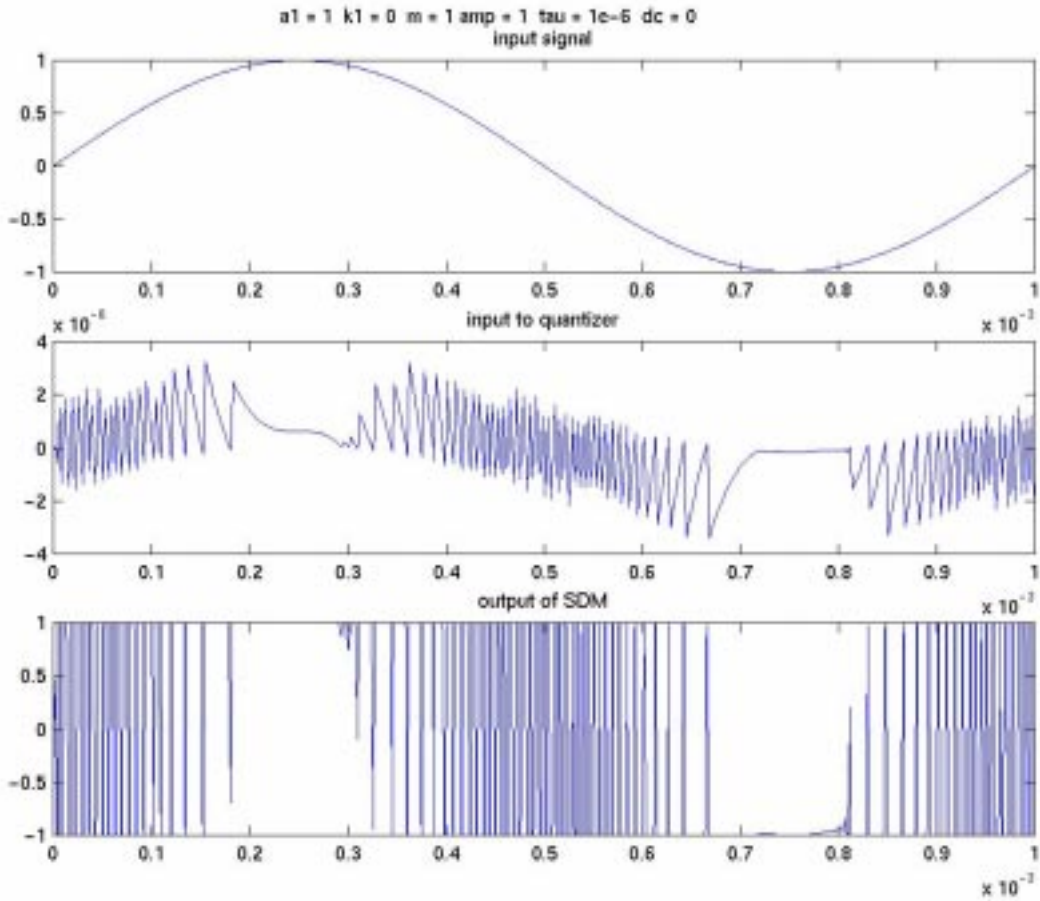


Figure 5: Ideal Integrator $k=0$. System is stable.

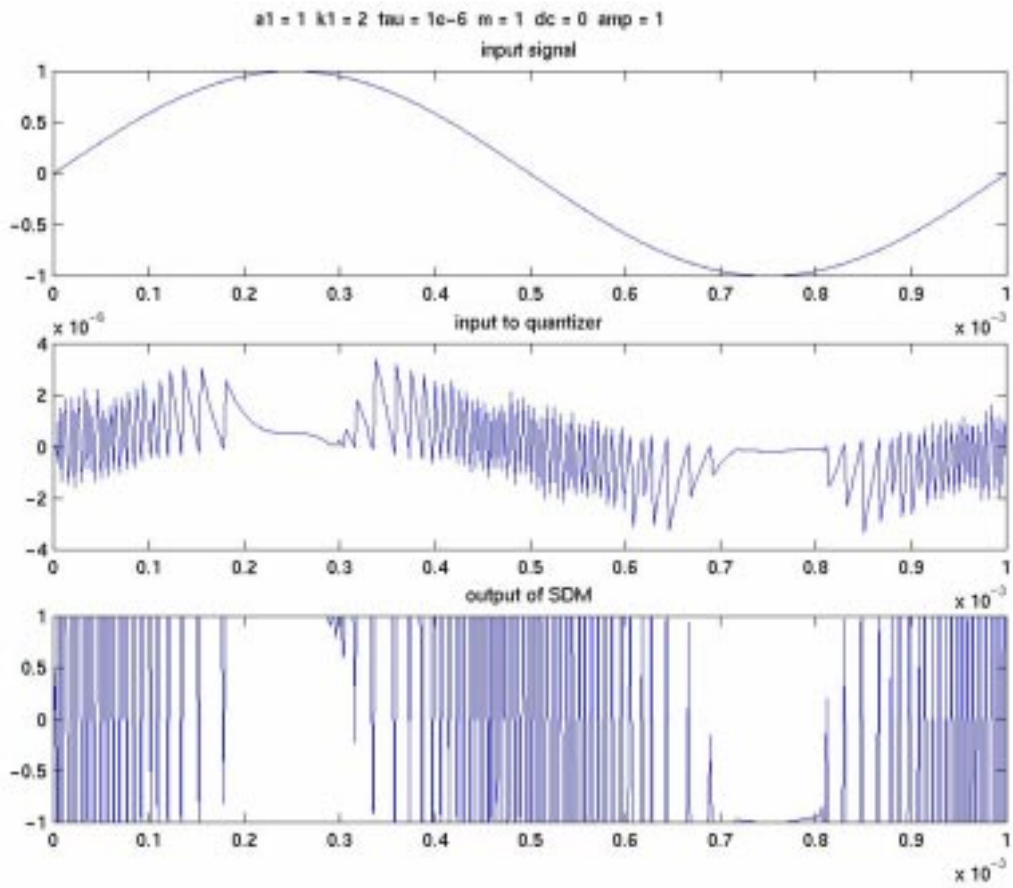


Figure 6: Integrator leakage (k_1) is increased to 2. System still stable, and the output of SDM is also accurate.

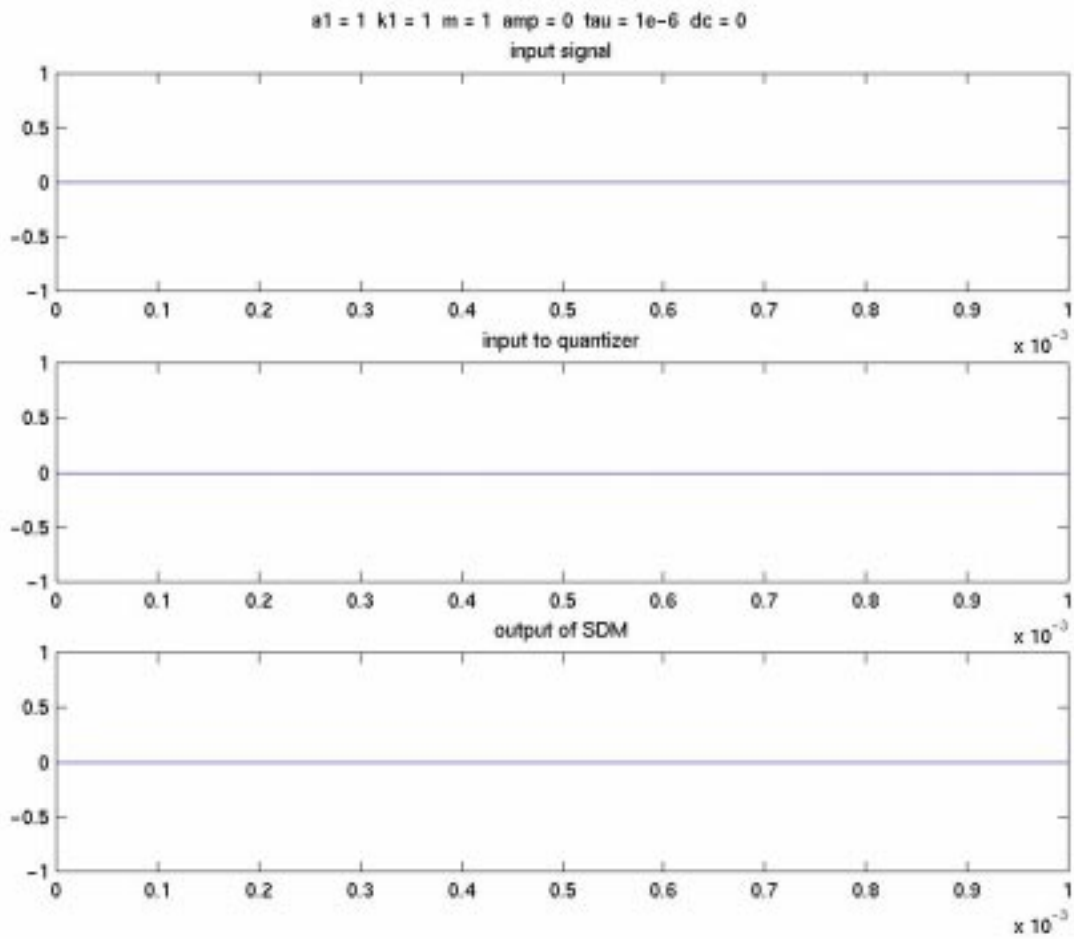


Figure 7: Input is 0 and the output of both Integrator and SDM is 0. System is stable and accurate.

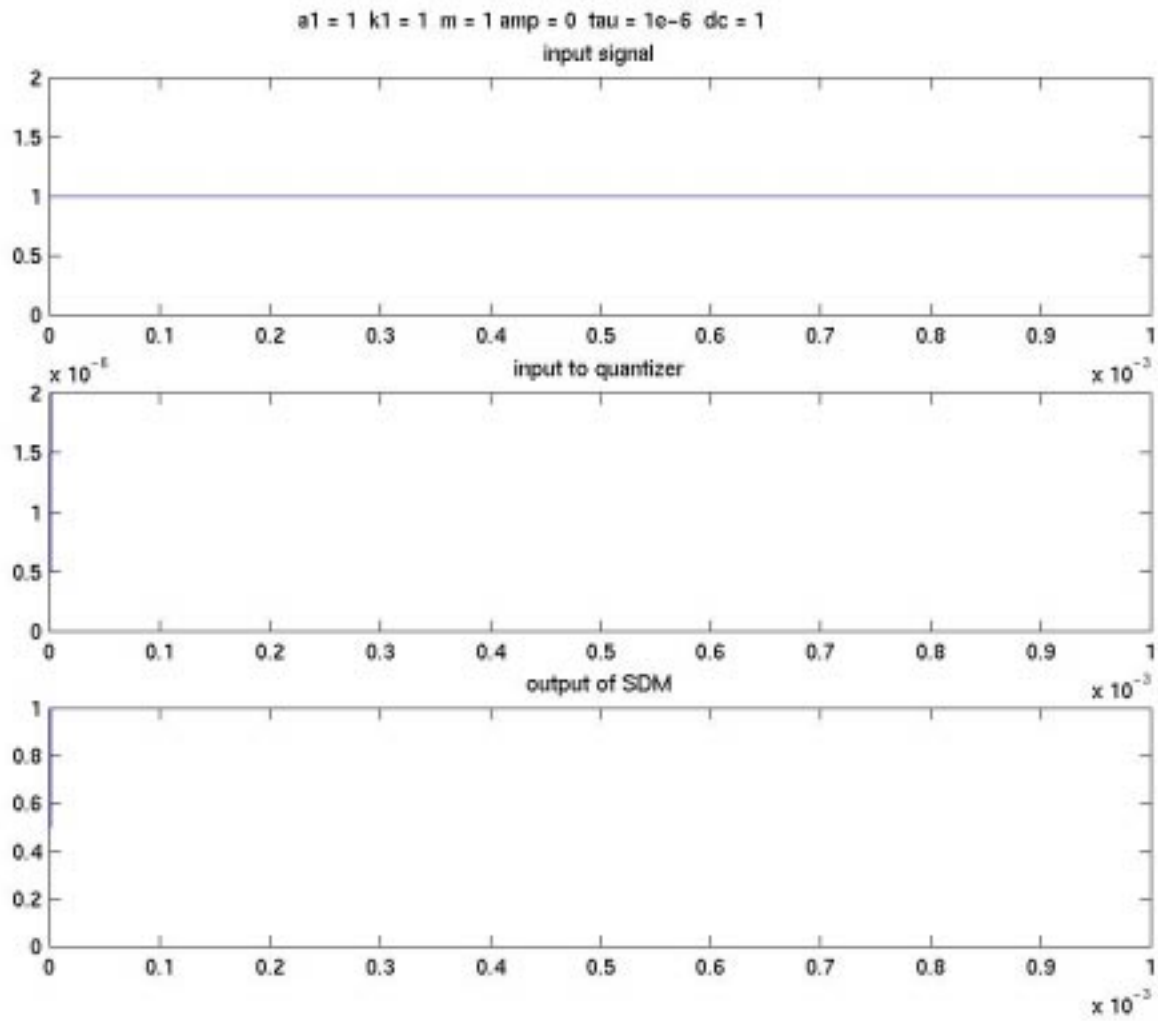


Figure 8: For dc input of 1, the system is stable and the output of SDM is accurate.

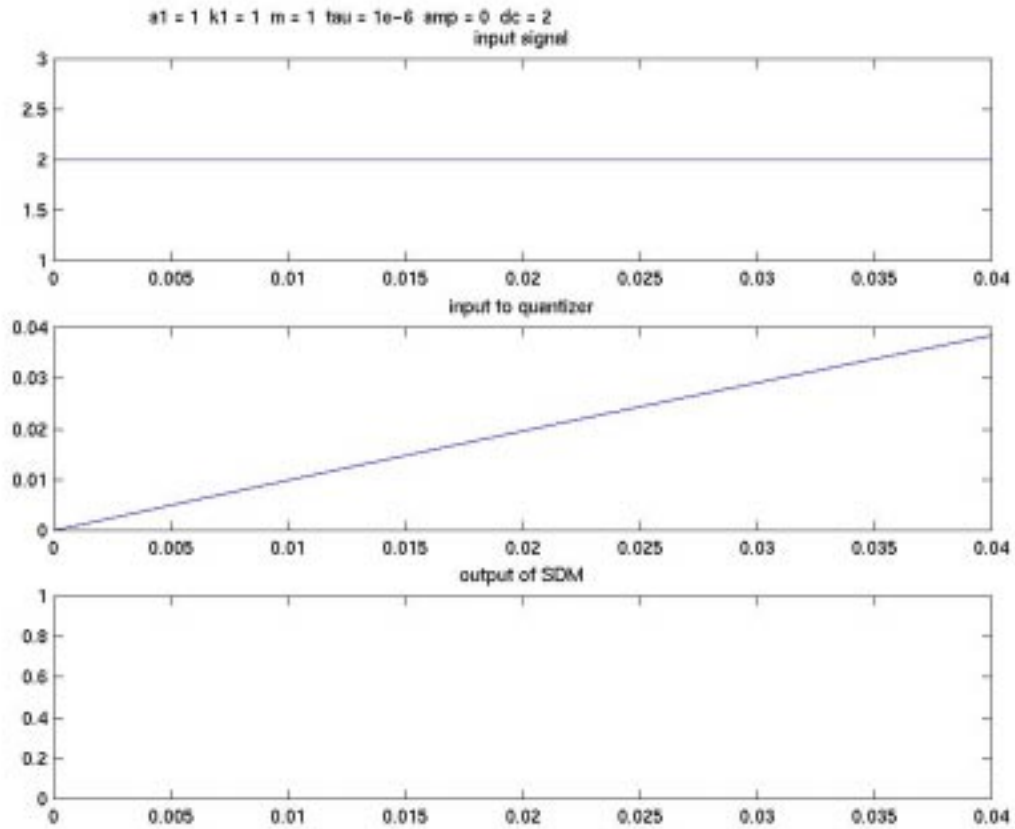


Figure 9: For dc input of 2, the output is accurate, but the input to quantizer is unbounded for 40ms, for a sampling frequency of 1Mhz. This indicates the possibility of unstable operation for a DC value greater than M.

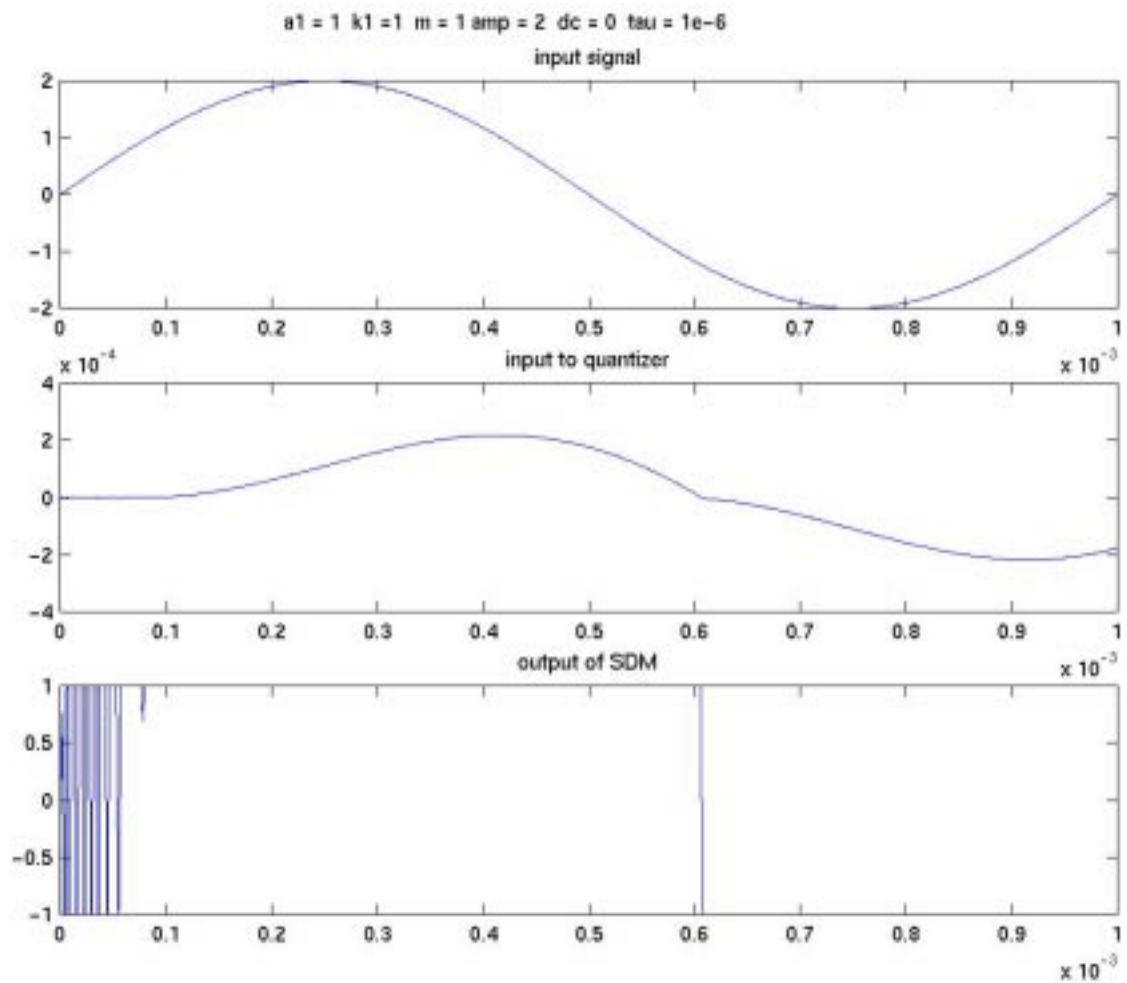


Figure 10: Amplitude of Sinusoidal input is 2. This causes possibility of inaccurate output of SDM, but appears stable at the input to the quantizer.

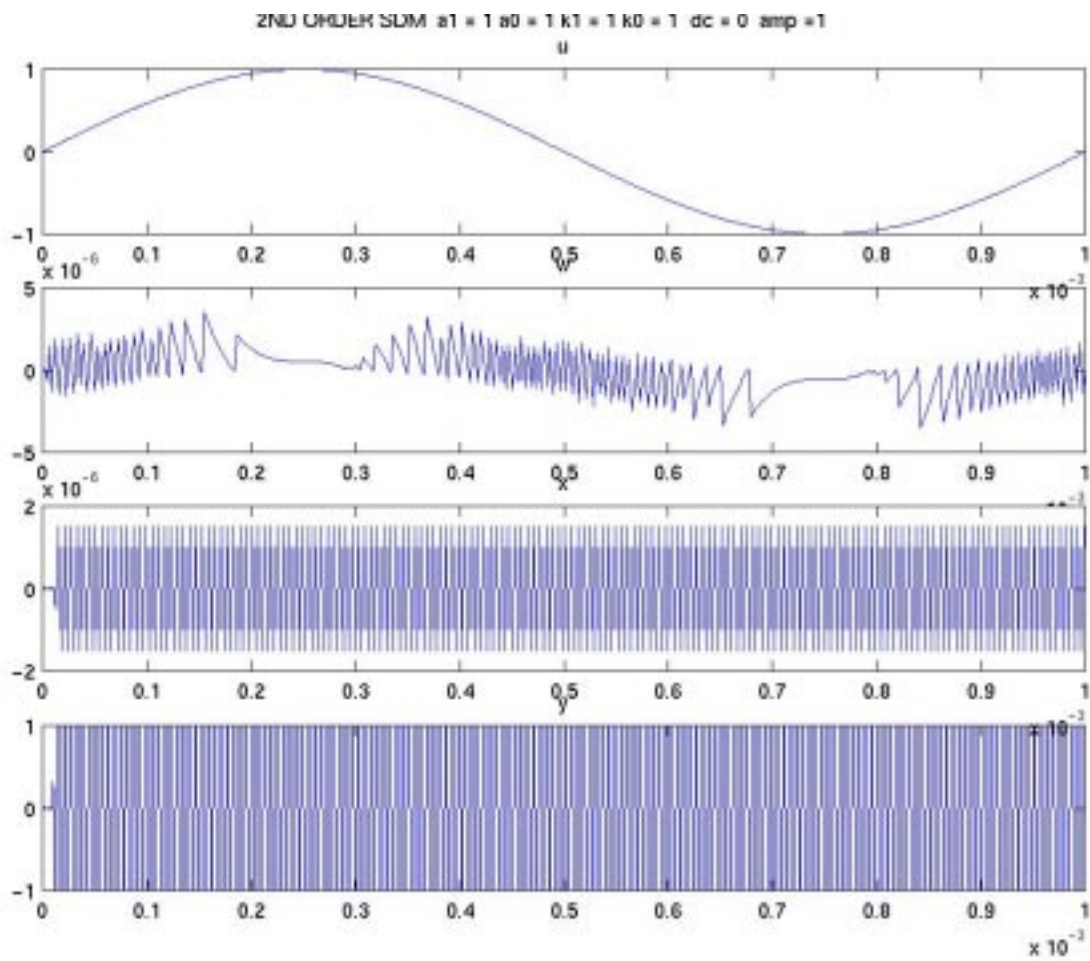


Figure 11: 2nd Order SDM with 1kHz Sinusoidal input. Stable.

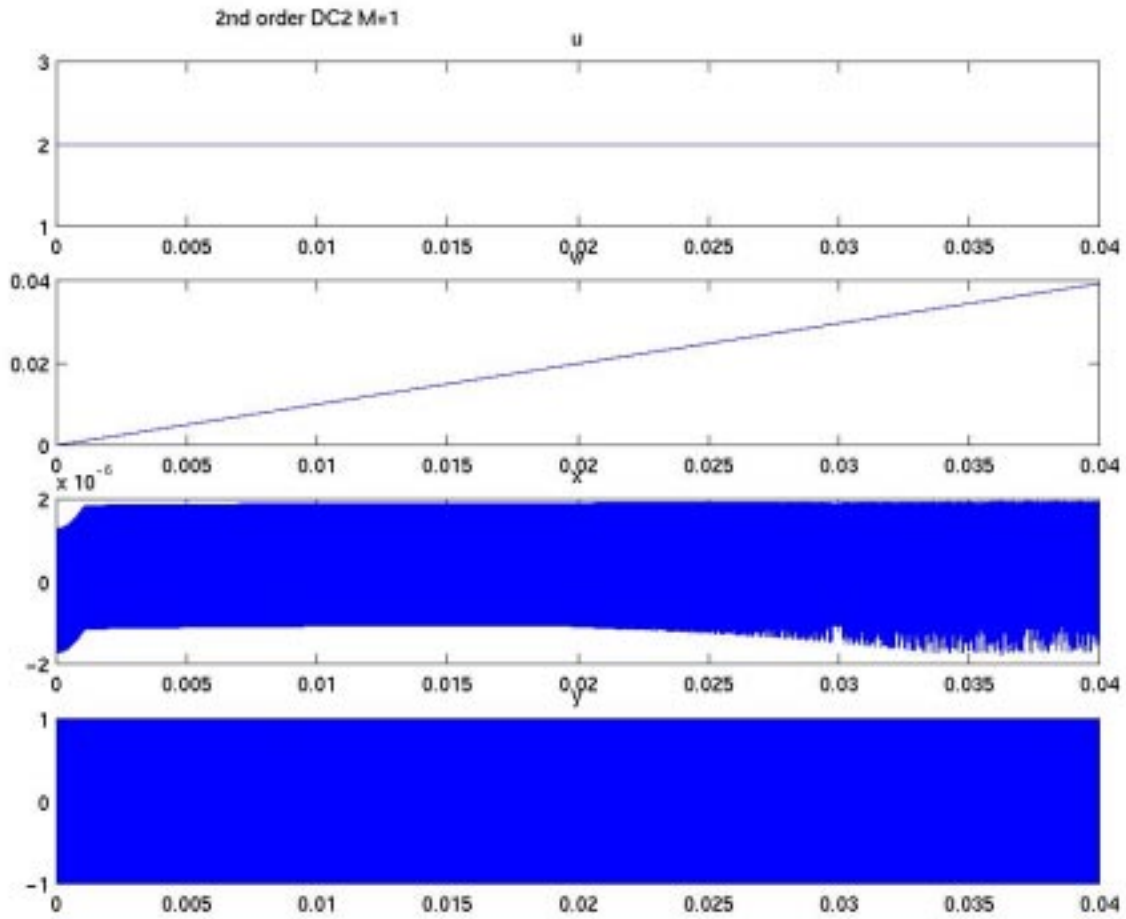


Figure 12: 2nd Order SDM with dc input greater than quantizer max.

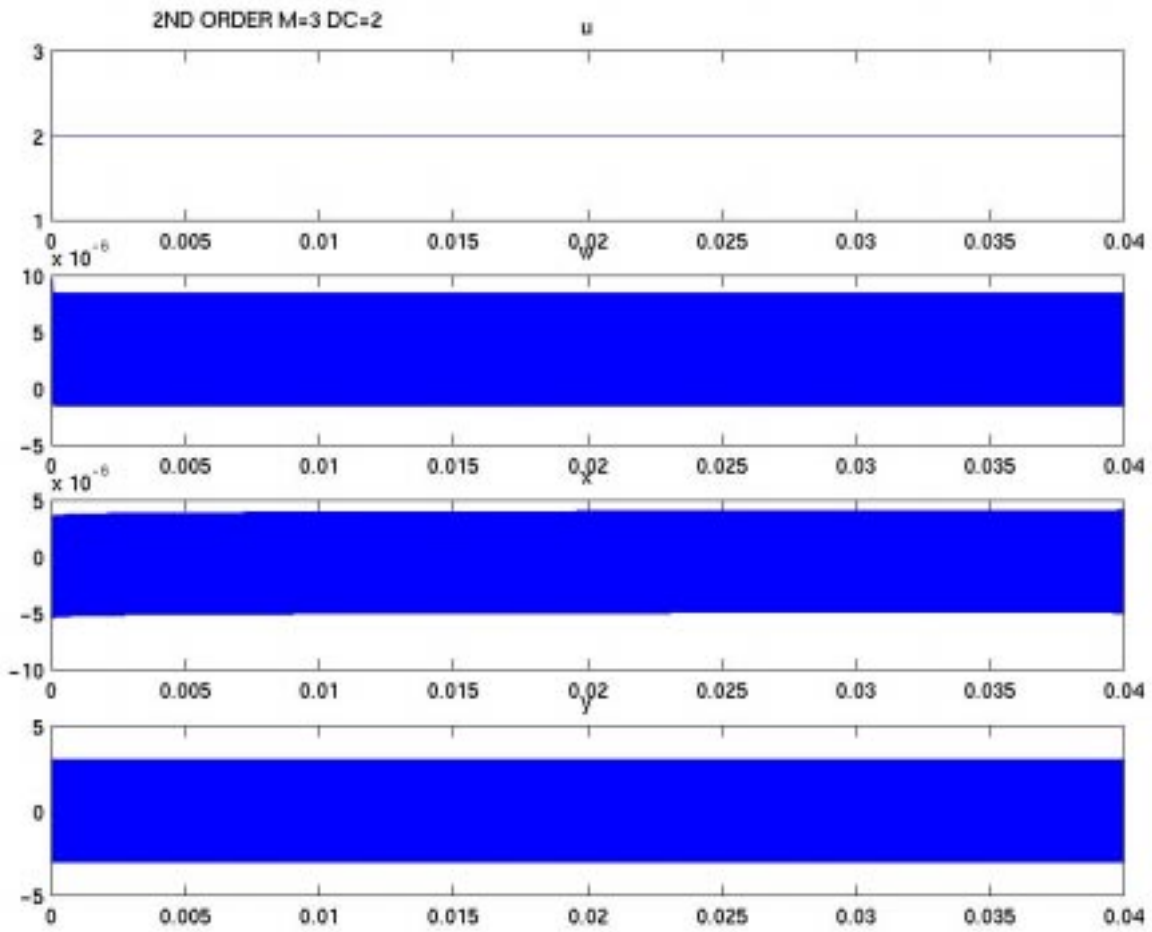


Figure 13: 2nd Order SDM with dc input less than quantizer max.

MATLAB CODE FOR 1ST ORDER SDM SOLVER

```
function [x,y,t] = odesolve(n)
%m decides the upper limit of tanh function, if m=5 => max. tanh=5
%p decides the slope of tanh
%k is the integrator leakage
%amp decides the amplitude of the Sinusoidal input wave above 2.5v DC
%dc is the dc level of the differential input signal, we will keep it 0
%x is the solution indicating the input to the quantiser
%t = Time 0 to lms
%increments used are lus, as the sampling frequency is 1Mhz
t0=0; %initial time
step=1e-6;
tau=step;
x(1)=0;
x(2)=0;
%n=number of samples;
dc=0;
k1=2;
a1=1;%integrator gain term
m=1;
p=2e7;
amp=1;
f=1000;

for i = 2:n
    t(i) = t0 + (i-1)*step ;

    x1=x(i-1);
    xc=x(i);

    An=step*deriv(t(i),xc,x1,m,p,k1,a1,amp,f,dc);

    temp_x=x(i)+An/2;

    Bn=step*deriv(t(i)+step/2,temp_x,x1,m,p,k1,a1,amp,f,dc);

    temp_x=x(i)+Bn/2;

    Cn=step*deriv(t(i)+step/2,temp_x,x1,m,p,k1,a1,amp,f,dc);

    temp_x=x(i)+Cn;

    Dn=step*deriv(t(i)+step,temp_x,x1,m,p,k1,a1,amp,f,dc);

    x(i+1)=x(i)+(1/6) * (An+2*Bn+2*Cn+Dn);

end;
t=[t0 t];

y=a1.* m.*tanh(p.*x./m);
figure;
subplot(3,1,3);
plot(t,y);
title('output of SDM');
```

```

subplot(3,1,2);
plot(t,x);
title('input to quantizer');

u=amp*sin(2*pi*f*t)+dc;
subplot(3,1,1);
plot(t,u);
title('input signal');
hold on;

```

MATLAB CODE FOR 2ND ORDER SDM SOLVER

```

function [x,y,w,t] = odesolvever2(n)
%used for solving 2 ODEs
%m decides the upper limit of tanh function, if m=5 => max. tanh=5
%p decides the slope of tanh
%k is the integrator leakage
%amp decides the amplitude of the Sinusoidal input wave above 2.5v DC
%dc is the dc level of the differential input signal, we will keep it 0
%x is the solution indicating the input to the quantiser

%t = Time 0 to 1ms
%increments used are 1us, as the sampling frequency is 1Mhz
t0=0; %initial time
step=1e-6;
tau=step;
x(1)=0;
x(2)=0;

w(1)=0;
w(2)=0;

%n=number of samples;
dc=0;
a0=1;%integrator gain term
a1=1;
k0=1;
k1=1;
m=1;
p=2e7;
amp=1;
f=1000;

for i = 2:n
    t(i) = t0 + (i-1)*step ;

    xl=x(i-1);
    xc=x(i);

    wl=w(i-1);
    wc=w(i);

```

```

An=step*deriv(t(i),wc,wl,m,p,k1,a1,amp,f,dc);
An1=step*deriv1(w(i),xc,xl,m,p,k0,a0);
temp_w=w(i)+An/2;
temp1_x=x(i)+An1/2;
Bn=step*deriv(t(i)+step/2,temp_w,wl,m,p,k1,a1,amp,f,dc);
Bn1=step*deriv1(temp_w,temp1_x,xl,m,p,k0,a0);
temp1_x=x(i)+Bn1/2;
temp_w=w(i)+Bn/2;
Cn=step*deriv(t(i)+step/2,temp_w,wl,m,p,k1,a1,amp,f,dc);
Cn1=step*deriv1(temp_w,temp1_x,xl,m,p,k0,a0);
temp1_x=x(i)+Cn1;
temp_w=w(i)+Cn;
Dn=step*deriv(t(i)+step,temp_w,wl,m,p,k1,a1,amp,f,dc);
Dn1=step*deriv1(temp_w,temp1_x,xl,m,p,k0,a0);
x(i+1)=x(i)+(1/6) * (An1+2*Bn1+2*Cn1+Dn1);
w(i+1)=w(i)+(1/6) * (An+2*Bn+2*Cn+Dn);

end;
t=[t0 t];
u=amp*sin(2*pi*1000*t)+dc;
figure;
y=a0.* m.*tanh(p.*x./m);
subplot(4,1,4);
plot(t,y);
title('Y');

subplot(4,1,2);
plot(t,w);
title('w');

subplot(4,1,3);
plot(t,x);
title('x');

subplot(4,1,1);
plot(t,u);
title('u');

hold on;

```

MATLAB FUNCTION USED IN 1ST AND 2ND ORDER SDM SOLVER

```
function z = deriv(ti,xc,xl,m,p,k1,a1,amp,f,dc)
%m decides the upper limit of tanh function, if m=5 => max. tanh=5
%p decides the slope of tanh at t=0
%k is the integrator leakage
%amp decides the amplitude of the Sinusoidal input wave above 2.5v DC
%dc is the dc level of the differential input signal, we will keep it 0
%x is the solution indicating the input to the quantiser

u=dc+amp*sin(2*pi*f*ti);

y=a1* m*tanh(p*xl/m);

w=k1*xc;

z=u-w-y;
```

MATLAB FUNCTION USED IN 2ND ORDER SDM SOLVER

```
function z1 = deriv1(w,xc,xl,m,p,k0,a0)

%solving 2nd loop

%m decides the upper limit of tanh function, if m=5 => max. tanh=5
%p decides the slope of tanh at t=0
%x is the solution indicating the input to the quantiser

y=a0* m*tanh(p*xl/m);

temp=k0*xc;

z1=w-temp-y;
```