

## Negative refraction of modulated electromagnetic waves

D. R. Smith<sup>a)</sup> and D. Schurig

*Physics Department, University of California, San Diego, La Jolla, California 92093*

J. B. Pendry

*The Blackett Laboratory, Imperial College, London, SW7 2BU, United Kingdom*

(Received 3 June 2002; accepted 12 August 2002)

We show that a modulated Gaussian beam undergoes negative refraction at the interface between a positive and negative refractive index material. While the refraction of the beam is clearly negative, the modulation interference fronts are not normal to the group velocity, and thus exhibit a sideways motion relative to the beam—an effect due to the inherent frequency dispersion associated with the negative index medium. In particular, the interference fronts appear to bend in a manner suggesting positive refraction, such that for a *plane wave*, the true direction of the energy flow associated with the refracted beam is not obvious. © 2002 American Institute of Physics.

[DOI: 10.1063/1.1512828]

Since the experimental demonstrations of a structured metamaterial with simultaneously negative permittivity and negative permeability<sup>1,2</sup>—often referred to as a *left-handed material*—the phenomenon of negative refraction, predicted to occur in left-handed materials,<sup>3,4</sup> has become of increasing interest. The consequence of a modulated plane wave incident on an infinite half space of negative refractive index material has been considered by Valanju, Walser, and Valanju,<sup>5</sup> who have noted that the interference fronts due to the modulation appear to undergo positive refraction, despite the underlying negative refraction of the component waves. Valanju and co-workers further associate the direction of the advancing interference pattern with that of the group velocity, and draw a broad set of conclusions regarding the validity of negative refraction.

In this letter, we examine the effect of a finite-width beam refracting at the interface between two semi-infinite media, one with positive refractive index and the other with negative refractive index. We find that the transmitted beam does indeed refract in the negative direction, although the interference fronts do not necessarily point in the same direction. We first describe the refraction of a modulated plane wave incident on a half space with negative index, which demonstrates in a straightforward manner that the interference fronts of the transmitted wave display interesting behavior.

We assume the geometry shown in Fig. 1, in which a modulated electromagnetic wave is incident from vacuum onto a material with frequency dependent, negative index of refraction,  $n(\omega)$ . We choose the  $z$  axis normal to the interface, the  $x$  axis in the plane of the figure, and the  $y$  axis out of the plane of the figure. While our arguments will apply to a wave with arbitrary polarization, we assume here that the electric field is polarized along the  $y$  axis (out of plane). For convenience, we use identical causal plasmonic forms for the permittivity ( $\epsilon$ ) and the permeability ( $\mu$ ), so that the refractive index is given by

$$n(\omega) = \sqrt{\epsilon\mu} = 1 - \frac{\omega_p^2}{\omega(\omega + i\Gamma)}. \quad (1)$$

Note that the refractive index is negative below the plasma frequency  $\omega_p$ , and has the value of approximately  $-1$  when  $\omega = \omega_p/\sqrt{2}$  and losses are small,  $\Gamma/\omega \ll 1$ .

A modulated plane wave has the form

$$E(\mathbf{r}, t) = e^{i\mathbf{k}\cdot\mathbf{r}} e^{-i\Omega t} \cos(\Delta\mathbf{k}\cdot\mathbf{r} - \Delta\omega t), \quad (2)$$

where  $\Omega$  is the carrier frequency of the wave,  $\Delta\omega$  is the modulation frequency and the wave vector,  $\mathbf{k}$ , is given in terms of the incident angle

$$\mathbf{k} = \sin(\theta)k\hat{\mathbf{x}} + \cos(\theta)k\hat{\mathbf{z}}, \quad (3)$$

where  $k = \omega/c$ . Equation (3) can be equivalently viewed as the superposition of two plane waves with different frequencies, or

$$E(\mathbf{r}, t) = \frac{1}{2} e^{i\mathbf{k}_+\cdot\mathbf{r}} e^{-i\omega_+ t} + \frac{1}{2} e^{i\mathbf{k}_-\cdot\mathbf{r}} e^{-i\omega_- t}, \quad (4)$$

where the frequencies of the component waves are  $\omega_{\pm} = \omega \pm \Delta\omega$  with corresponding propagation vectors  $\mathbf{k}_{\pm} = \mathbf{k} \pm \Delta\mathbf{k}$ .  $\Delta\mathbf{k}$  is parallel to  $\mathbf{k}$  and is given by

$$\Delta\mathbf{k} = \frac{\Delta\omega}{\omega} \mathbf{k}. \quad (5)$$

Because the negative index medium is assumed to be frequency dispersive, the transmitted waves will refract at slightly different angles, thus introducing a difference in the phase versus interference propagation directions. The transmitted wave can be expressed in the form

$$E(\mathbf{r}, t) = e^{i\mathbf{q}\cdot\mathbf{r}} e^{-i\Omega t} \cos(\Delta\mathbf{q}\cdot\mathbf{r} - \Delta\omega t), \quad (6)$$

which again is composed of two propagating plane waves having propagation vectors  $\mathbf{q}_+$  and  $\mathbf{q}_-$ , the values of which can be found using the conservation of the parallel wave vector across the interface

$$q_x = k_x \quad (7)$$

and the constancy of frequency

<sup>a)</sup>Electronic mail: drs@sdss.ucsd.edu

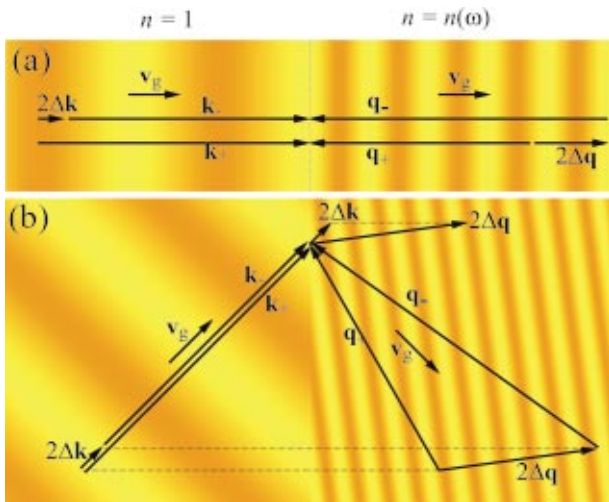


FIG. 1. (Color) Normal incidence (a). The group velocity and interference wave vector are parallel. Incidence at 45° angle (b). On the incident side (left) the component wave vectors, interference wave vector and group velocity all point in the same direction. In the NIM (right) the interference wave vector has refracted positively and the component wave vectors and group velocity have refracted negatively with the component wave vectors approximately antiparallel to the group velocity. The dashed lines indicate conservation of the transverse component of the various wave vectors.

$$\mathbf{q} \cdot \mathbf{q} = n^2 k^2 \tag{8}$$

resulting in

$$\mathbf{q} = k_x \hat{\mathbf{x}} \pm \sqrt{n^2 k^2 - k_x^2} \hat{\mathbf{z}}. \tag{9}$$

This expression for individual plane waves is equivalent to Snell’s law in predicting the direction of refraction. The correct sign of the square root is well established elsewhere.<sup>6,7</sup> Using  $\Delta q_x = \Delta k_x$  and Eq. (8), the wave vector of the interference pattern is obtained

$$\Delta \mathbf{q} = \left[ k_x \hat{\mathbf{x}} + \frac{1}{q_z} (k^2 n n_g - k_x^2) \hat{\mathbf{z}} \right] \frac{\Delta \omega}{\Omega}. \tag{10}$$

The group index, given by

$$n_g \equiv \frac{\partial(n\omega)}{\partial\omega} \geq 1 \tag{11}$$

is constrained by causality to be greater than unity regardless of the sign of the refractive index of the medium. Thus, we see that for a negative refractive index the  $z$  component of the interference wave vector has an opposite sign to the  $z$  component of the individual wave vectors, so though the individual wave vectors refract negatively, the interference pattern refracts positively. Furthermore, the apparent velocity of the interference fronts is directed away from the interface, as would be expected if the direction of the interference pattern were coincident with the direction of the group velocity. Note that Eq. (10) also implies that the spacing between interference fronts (or beats) in the negative index medium is set by the dispersion characteristics of the medium rather than just the frequencies of the two component waves. This effect is illustrated in Fig. 1(a) for a wave normally incident from vacuum on a medium with a frequency dispersive negative refractive index. In this example, the carrier frequency is assumed to be at the frequency where  $n = -1$ . For our choice of frequency dispersion [Eq. (1)],  $\partial(n\omega)/\partial\omega = 3$ , and

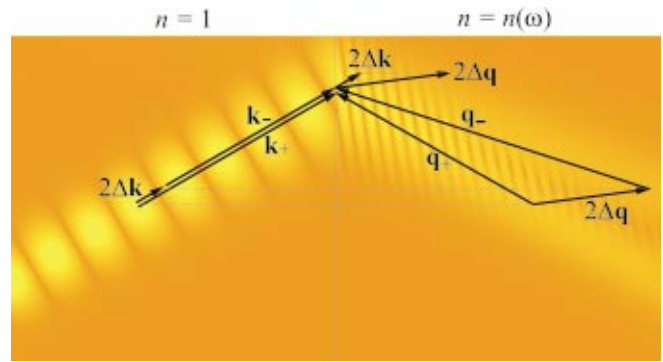


FIG. 2. (Color) Refraction of a Gaussian beam into a NIM. The angle of incidence is 30°,  $n(\omega_-) = -1.66 + 0.003i$  and  $n(\omega_+) = -1.00 + 0.002i$ .  $\Delta\omega/\omega = 0.07$ .

Eq. (10) shows that the spacing between interference fronts is one third that in free space, as seen in the figure.

When a monochromatic wave is incident on the interface between positive and negative media at an angle, previous calculations and theory<sup>1,3</sup> have shown that the phase velocity is negative, and the transmitted ray in the medium is directed toward the interface and negatively refracted in accordance with Snell’s law. Equation (10), however, indicates that the interference fronts associated with a modulated plane wave should undergo positive refraction. A calculation of the refraction of a modulated plane wave shows that this is indeed the case [Fig. 1(b); see also EPAPS<sup>8</sup>]: the incident interference pattern refracts to the opposite side of the normal, consistent with Valanju and co-workers.<sup>5</sup>

The question thus naturally arises: what is the relationship between the interference fronts associated with a modulated wave and its group velocity? The group velocity is defined as<sup>9</sup>

$$\mathbf{v}_g \equiv \nabla_{\mathbf{q}} \omega(\mathbf{q}), \tag{12}$$

which, for isotropic, low loss materials becomes

$$\mathbf{v}_g = \frac{\mathbf{q}}{q} \frac{d\omega(q)}{d\omega} = \frac{\mathbf{q}}{q} \text{sign}(n) \frac{c}{n_g}. \tag{13}$$

The scalar part of this expression is a material property and the unit vector pre-factor is determined by the direction of the incoming wave. Note that when the index is negative, Eq. (13) predicts that the direction of the group velocity is antiparallel to the phase velocity, and thus refracts negatively—unlike the interference wave vector. This would seem to be a paradox, as the interference velocity of a modulated beam is frequently equated with the group velocity.<sup>9</sup> To resolve this dilemma, we examine the manner in which a point of constant phase on the interference patterns moves. Setting the argument of the cosine in Eq. (6) equal to a constant, we find that the velocity of such a point,  $\mathbf{v}_{\text{int}}$ , must satisfy

$$\frac{\Delta \mathbf{q}}{\Delta \omega} \cdot \mathbf{v}_{\text{int}} = 1. \tag{14}$$

The smallest possible such velocity is the one parallel to  $\Delta \mathbf{q}$  (the interference direction), which has been interpreted as the group velocity by Valanju and co-workers;<sup>5</sup> however,  $\mathbf{v}_{\text{int}}$  can have an arbitrary component perpendicular to  $\Delta \mathbf{q}$ . For non-normal incidence, the group velocity has such a component.

Using Eqs. (10) and (13), we can verify that the group velocity does indeed satisfy Eq. (14). We thus conclude that a *component* of the group velocity follows the interference pattern, but that there is also a component *parallel* to the interference wave fronts.

Since we have concluded that the interference velocity need not be in the same direction as the group velocity, we cannot easily ascertain the group velocity from the field pattern of the transmitted part of a modulated plane wave refracting at the interface of a negative index material. To obtain a better picture of the refraction phenomenon, we consider a modulated beam of finite width. Following the method outlined by Kong, Wu, and Zhang,<sup>10</sup> we calculate the refraction associated with an incident modulated beam with a finite (Gaussian) profile. The electric field as a function of position is determined from the inverse Fourier transform of the  $k$ -space field

$$E(x, z) = F^{-1}\{E(k_x, z)\} = \int_{-\infty}^{+\infty} dk_x e^{ik_x x} E(k_x, z), \quad (15)$$

where the  $k$ -space field is determined by solving the boundary value problem

$$E(k_x, z) = E_i(k_x) \begin{cases} e^{ik_z z} + \rho e^{-ik_z z} & z < 0 \\ \tau e^{iq_z z} & z > 0 \end{cases} \quad (16)$$

and the expression

$$E_i(k_x) = \frac{g}{2\sqrt{\pi}} e^{-(1/4)g^2(k_x - k_{xc})^2} \quad (17)$$

provides the Gaussian shape of the incident beam. The incident beam is centered about the wave vector with  $k_x = k_{xc}$ . When the wave interacts with the interface, reflected and transmitted waves are generated. Expressions for the reflection and transmission coefficients are determined by matching the electric and magnetic fields at the interface between the negative and positive index media. The transmission coefficient is

$$\tau = \frac{2\mu k_z}{\mu k_z + q_z} \quad (18)$$

and the reflection coefficient is

$$\rho = \frac{\mu k_z - q_z}{\mu k_z + q_z}. \quad (19)$$

To produce a modulated beam, we superpose two beams with different frequencies, and hence different incident wave vectors  $\mathbf{k}_c$ .

Figure 2 shows the intensity pattern of the electric field in a beam undergoing refraction and reflection at the interface between vacuum and a negative refractive index material. The modulation is evident in the spacing between intensity maxima (the carrier wave is not shown in the figure; see also EPAPS<sup>8</sup>).

Although the interference fronts associated with the modulation are not perpendicular to the direction of the refracted beam, the modulation is indeed carried through. As would be expected, as the time step is advanced the transmitted modulation fronts advance in the direction of the beam, even though the phase velocity of the carrier is antiparallel to the beam propagation direction. This behavior is both consistent with the notion of negative refraction, as well as the results obtained by Valanju and co-workers.<sup>5</sup>

Negative refractive materials are necessarily frequency dispersive, so that the various frequency components of a modulated beam are refracted at different angles within the medium. This effect is made clear in Fig. 2, where further from the interface, within the medium, the two component beams have separated, and are also damped by the losses included in the calculation. Nevertheless, were the beam to interact with a finite section of negative index material, the modulation would be preserved in the transmitted beam. Note also that we have simulated very large modulation frequencies in these simulations to demonstrate these effects—much larger than would be necessary for the transfer of information in typical communications applications.

Through an analysis of the points of constant phase of a modulated plane wave, we have shown that both the group and phase velocities undergo negative refraction at the interface between a positive and a negative index material. The interference fronts of the modulated wave are not *normal* to the group velocity, and exhibit a sideways motion as they move *at* the group velocity. Consideration of a modulated beam of finite extent clearly resolves the difference between the group velocity and normal to the interference fronts, and shows that recent observations<sup>5</sup> are consistent with negative refraction.<sup>3</sup> These conclusions hold also for certain bands in photonic band gap structures that exhibit negative effective refractive index.<sup>11,12</sup>

The authors thank Professors S. Schultz and N. Kroll for valuable discussions. This work was supported by DARPA through a grant from ONR (Contract No. N00014-00-1-0632).

<sup>1</sup>D. R. Smith, W. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, Phys. Rev. Lett. **84**, 4184 (2000).

<sup>2</sup>R. A. Shelby, D. R. Smith, and S. Schultz, Science **292**, 79 (2001).

<sup>3</sup>V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968).

<sup>4</sup>D. R. Smith and N. Kroll, Phys. Rev. Lett. **85**, 3966 (2000).

<sup>5</sup>P. M. Valanju, R. M. Walser, and A. P. Valanju, Phys. Rev. Lett. **88**, 187401 (2002).

<sup>6</sup>J. B. Pendry, Phys. Rev. Lett. **85**, 3966 (2000).

<sup>7</sup>R. W. Ziolkowski and E. Heyman, Phys. Rev. E **64**05, 056625 (2001).

<sup>8</sup>See EPAPS Document No. E-APPLAB-81-022240 for (brief description). A direct link to this document may be found in the online article's HTML reference section. The document may also be reached via the EPAPS homepage (<http://www.aip.org/pubservs/epaps.html>) or from <ftp.aip.org> in the directory /epaps/. See the EPAPS homepage for more information.

<sup>9</sup>M. Born and E. Wolf, *Principles of Optics* (Cambridge University Press, Cambridge, UK, 1980), Chap. I.

<sup>10</sup>J. A. Kong, B. Wu, and Y. Zhang, Appl. Phys. Lett. **80**, 2084 (2002).

<sup>11</sup>M. Notomi, Phys. Rev. B **62**, 10696 (2000).

<sup>12</sup>B. Gralak, S. Enoch, and G. Tayeb, J. Opt. Soc. Am. A **17**, 1012 (2000).