Electromagnetic source transformations using superellipse equations

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Transformation optics can be used to design media with unique properties that alter the behavior of electromagnetic waves in passive space and recently in space containing source distributions. We present source transformations where current from a linear radiator is spread over a cylindrical shell with various cross sections. The semianalytic transformations are based on superellipse equations. Finite-element full-wave simulations of transformations from a dipole to a cylinder, diamond-shaped cylinder, and flattened cylinder are presented. The radiation pattern of the dipole seen by an outside observer is replicated in all cases demonstrating the potential applicability of source transformations to conformal antenna design.
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Transformation optics is a methodology that facilitates the design of complex electromagnetic structures and devices. The transformation optical approach has gained attention as a path to designing media that can "cloak" objects from detection by electromagnetic or other waves.[1–3] In a cloaking structure, passive free space is transformed into a new space having different spatial and material characteristics.[1] This transformation is carried out by introducing a singularity (concealed region), which maps to the inner surface of the cloaking structure.[4] The cloaking transformation is one that can inherently be modified to include source distributions since it is possible to impress an arbitrary current or charge distribution upon the singular region. Source transformations thus represent an extension of the transformation optical technique and can be applied to a variety of practical antenna schemes.

The concept of source transformations was recently considered by Luo *et al.*,[5] who showed that a line current carried by a dipole antenna could be mapped to a surface current on the inner boundary of a spherical shell. Thus, the radiation pattern of the original line dipole in free space (from the perspective of an outside observer) was reproduced by the spherical current-carrying shell surrounded by the properly designed transformation optical medium. In a different context, the use of source transformations to alter the radiation pattern of arbitrary-shaped antennas was considered by Kundtz *et al.*[6] The use of source transformations provides interesting opportunities for the design of radiating structures. Here, we present examples that combine current density transformation with corresponding transformation media using superellipse equations to illustrate the wide applicability of the source transformation methodology. The superellipse equations can be used to describe different shapes with a single set of mathematical functions and are thus ideal to demonstrate how source transformations can be applied to conformal antennas of varied geometry by surrounding sources with transformation media.

The electric and magnetic fields in transformation optics can be described by form-invariant Maxwell's equations.[7,8] The information about the spatial transformation is contained in the transformed material parameters and sources, thus a new set of curl and divergence equations does not have to be defined for each set of new coordinates. We perform the following steps to design an appropriate transformation. (i) We first determine a coordinate transformation from Cartesian coordinates (system 1) to a system (system 2) that facilitates the desired field behavior with the correct functional form. (ii) We then convert the coordinates from system 2 to the next coordinate system (system 3) that is used to describe the transformed space. (iii) We replace the transformed coordinates (system 3) by coordinates of system 2. This step maintains the forms of the fields, currents, and material parameters but associates them with the coordinates of system 2. (iv) Finally, we calculate the fields, currents, and material parameters by transforming the coordinates back to the original Cartesian coordinates (system 1).[6]

The material parameters are determined using the usual transformation optics approach with Maxwell's form-invariant equations. The permittivity and permeability can be calculated using $\varepsilon_r^{i'j'}=(1/|A|)A_i^{i'}A_j^{j'}\varepsilon_r^{ij}$ and $\mu_r^{i'j'}=(1/|A|)A_i^{i'}A_j^{j'}\mu_r^{ij}$, respectively, where $A_i^{i'}=\partial x^{i'}/\partial x^i$.[7] The current can be transformed as a vector density using $j'=(1/|A|)A_i^{i'}j^i$, where $A_i^{i'}$ has the same definition used in the transformation of the material parameters.[7] It is important to note that even in the source transformation, the information about the coordinate change is carried in the material parameters and current density. However, the total current carried by the original source is conserved.[6]

To describe the cross-sectional shape of the transformed structure, we describe the transformation analytically using the superellipse equation, which is extensively used for modeling a wide range of shapes in computer graphics.[9] In three-dimensional space, it has also been applied to rounded cuboids and rounded cylinder cloaks.[10] Cartesian coordinates can be expressed parametrically in two dimensions using the superellipse equations that are as follows: $x=|\cos(\phi)|^{2/n}\cdot a\cdot\mathrm{sign}[\cos(\phi)]$ and $y=|\sin(\phi)|^{2/m}\cdot b\cdot\mathrm{sign}[\sin(\phi)]$, where $m,n>0$. By changing the powers (m,n) and the ratio of the scaling factors (a,b), a

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variety of geometrical shapes can be described. This superellipse formula holds the promise of transforming space with a variety of different shapes using the same mathematical basis, thus providing a straightforward method to design devices such as conformal antennas using transformation optics.

Our goal is to apply a source transformation such that a simple dipole in free space can be replaced by an appropriately designed metamaterial shell (with a current distributed on the inner surface) of a shape designed using the superellipse equation. Similar to the invisibility cloak demonstrated by Pendry *et al.*[1] and the source transformation published by Luo *et al.*,[5] a and b in the parametric equations for x and y are set to $a=[R_2/(R_2-R_1)]\cdot(r-R_1)$ and $b=[(R_2-\alpha)\cdot(r-R_1)/(R_2-R_1)]+\alpha$, where $\alpha=h/2$ and r is valid from $R_1 \le r \le R_2$. This transformation compresses the space contained in a volume of radius R_1 into a shell of inner radius R_1 and outer radius R_2, where h is the length of the antenna.[5] Note that this transformation takes a singular region that has the form of a line at the origin stretching from $y=-h/2$ to $y=+h/2$, which can be viewed as a linear antenna and spreads it over the inner surface of the transformed cylinder.

The Jacobian for the transformation from the Cartesian coordinates to the coordinate system that facilitates the desired field behavior (system 2) with the correct functional form is

$$A_1=\begin{pmatrix} \cos(\phi)^{2/n} & 2r\cdot\cos(\phi)^{-1+2/n}\sin(\phi)/n & 0 \\ \sin(\phi)^{2/m} & 2r\cdot\cos(\phi)\sin(\phi)^{-1+2}/m & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the Jacobian for the transformation from coordinate system 2 to the coordinate system that dictates desired field behavior (system 3) form is

$$A_2=\begin{pmatrix} [R_2\cos(\phi)^{2/n}]/(R_2-R_1) & -[2a\cdot\cos(\phi)^{-1+2/n}\sin(\phi)]/n & 0 \\ \left(R_2-\dfrac{h}{2}\right)\cdot\sin(\phi)^{2/m}/(R_2-R_1) & [2r\cdot b\cdot\cos(\phi)\sin(\phi)^{-1+2/m}]/m & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$j=\begin{pmatrix} 0 \\ 1/(\sqrt{\sigma\cdot\pi})e^{-(x)^2/\sigma} \\ 0 \end{pmatrix}$$

is used to approximate the current distribution on a thin wire at $x=0$ by setting σ to be infinitesimal relative to the length of the antenna.

The current source and material transformations are simulated using a commercial finite-element based electromagnetics solver (COMSOL MULTIPHYSICS©). This software allows the specification of material anisotropy and continuous inhomogeneity. Two-dimensional, time-harmonic, full wave simulations were carried out with the polarization of the waves constrained to be transverse magnetic. At the inner boundary of the metamaterial shell, a perfect electric conductor boundary is created. At this boundary, an active magnetic field discontinuity is placed to create a tangential current density in each of the example geometries demonstrated here.

We present three examples of source transformations using the superellipse equation. In the first example, we show that a dipole with a constant current distribution in free space can be replaced by a cylindrical transformation medium with a constant current distribution on the inside boundary. It can be seen that the metamaterial shell can reproduce the field radiation pattern of the original dipole in free space as seen by an outside observer. The cylindrical medium was designed by using $m=n=2$ in the superellipse equations, where a and b are defined above. In the second example, a dipole with a constant current distribution in free space was replaced by a diamond-shaped cylinder. Here, $m=n=1$ in the superellipse equations. These examples are shown in Fig. 1. In the third example, a more arbitrary shape was chosen. A dipole with a constant current distribution operating in free space was replaced by a flattened cylinder. In this example $m=1$, $n=3$, and a and b are defined, as shown above, but a is six times larger than b to produce the axial asymmetry required to produce the shape (refer to Fig. 2). The simulation fields produced by all three examples closely approximate the original radiation pattern of the dipole.

The source transformations studied here (cylinder, diamond-shaped cylinder, and flattened cylinder) reveal the generality of the source transformation approach. The method should be applicable to the design of conformal antennas, given the goal in such systems of achieving a desired radiation pattern with an arbitrary current distribution distributed over a surface. The transformation optical medium that must be included in the system can be achieved utilizing properly designed metamaterials. As is typical with the transformation optical designs, the material properties of the resulting composite are highly anisotropic with values of the μ and ε tensor elements taking sometimes extreme values, especially near the inner radius of the metamaterial coatings. In addition, for the asymmetric structures, the orientation of the principal axes varies nontrivially throughout the medium. Far more reasonable ranges of values of the tensor components can be obtained through optimization of the transformation,[11] which was not attempted here. Given the

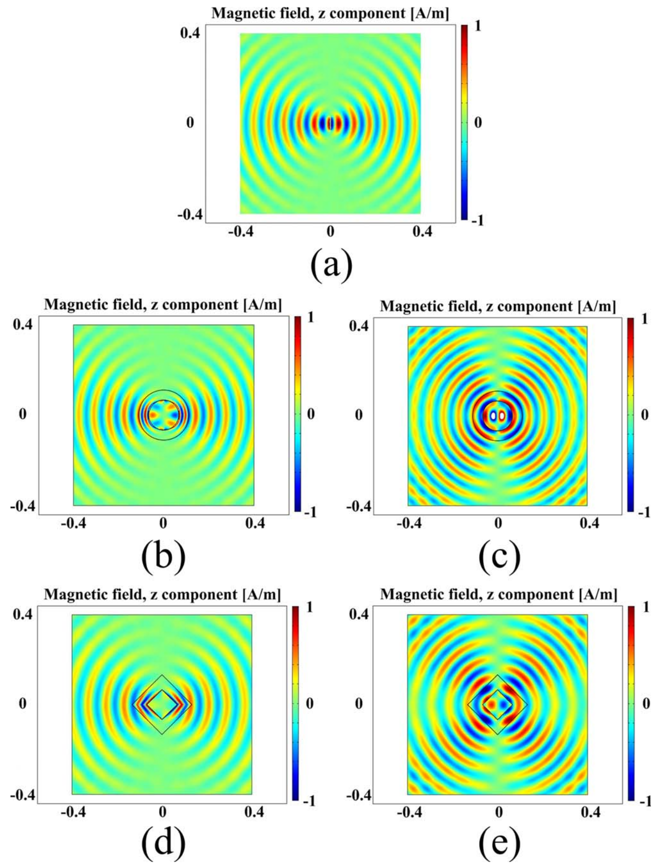

FIG. 1. (Color online) Radiation pattern of examples to illustrate source transformation method. The z-component of the magnetic field (H_z) for (a) the dipole in free space of length $h=\lambda$, where λ is the operating wavelength in free space, (b) the dipole in (a) to cylinder transformation, (c) the dipole in (a) to cylinder transformation without metamaterial, (d) the dipole in (a) to diamond-shaped cylinder transformation, and (e) the dipole in (a) to diamond-shaped cylinder transformation without metamaterial.

nature of the transformation, the driving point admittance of the sources used to form the radiating structures should not change from the original structure to the transformed structure, suggesting a potential advantage worthy of further pursuit in the context of antenna technology.

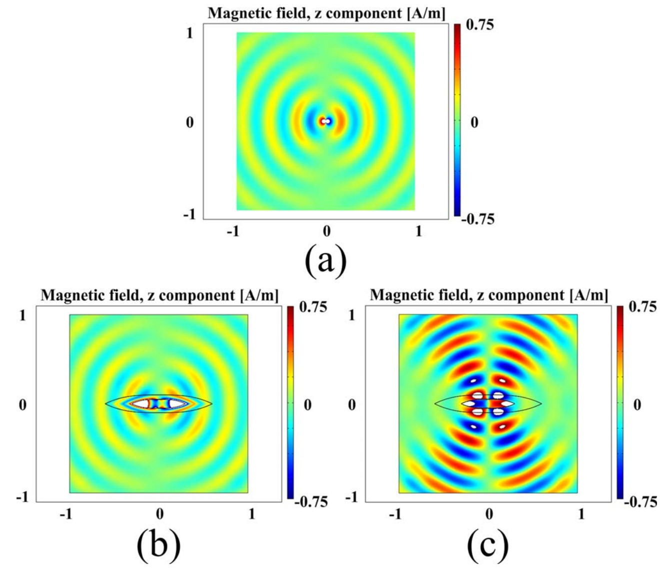

FIG. 2. (Color online) Radiation pattern represented by the z-component of the magnetic field (H_z) for (a) the dipole in free space of length $h=\lambda/4$, where λ is the operating wavelength in free space, (b) the dipole in (a) to flattened cylinder transformation, and (c) the dipole in (a) to flattened cylinder transformation without metamaterial.

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