A rigorous and nonsingular two dimensional cloaking coordinate transformation

Steven A. Cummer,[1,a)] Ruopeng Liu,[1] and Tie Jun Cui[2,b)]
[1]*Department of Electrical and Computer Engineering and Center for Metamaterials and Integrated Plasmonics, Duke University, Durham, North Carolina 27708, USA*
[2]*State Key Laboratory of Millimeter Waves, Southeast University, Nanjing 210096, People's Republic of China*

(Received 5 December 2008; accepted 14 January 2009; published online 4 March 2009)

We derive an approximate but rigorous two dimensional electromagnetic cloaking shell in which no parameters are singular. The simplicity of the material parameters makes it potentially easier to realize in practice than other cloaking solutions. The analytical formulation of this shell also provides a simple and explicit analytical trade-off between its scatter reduction performance, its thickness relative to the interior size, and the complexity and realization difficulty of the required permittivity and permeability. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3080155]

Pendry *et al.*[1] reported electromagnetically anisotropic and inhomogeneous shells that, in theory, completely shield an interior structure of arbitrary size from electromagnetic fields without perturbing the external fields. Numerical simulations[2] and analytical electromagnetic scattering theory[3] have confirmed that these shells behave as predicted by the original coordinate transformation theory.

One of the most remarkable aspects of this theory is that the number of material parameter sets required to realize a given application is essentially unbounded. The most studied cloaking transformation is based on a linear radial transformation reported in the initial paper on the subject.[1] References 4 and 5 explored a different two dimensional (2D) transformation in order to achieve, in the short wavelength limit, a unity value for one permeability component and free space values at the outer edge for TM incidence. There remains a large unexplored space of cloaking transformations that could yield improved practical performance or simplified physical realizability.

In this work we derive an approximate but rigorous 2D electromagnetic cloaking shell with parameters that are simpler to realize in practice than other formulations in that no parameters are singular and the in-plane permittivity and permeability components can be made spatially uniform. This is the first 2D cylindrical cloaking transformation for which singularities are rigorously removed without invoking a ray-tracing approximation. This cloaking shell also provides a simple analytical tradeoff between its scatter reduction performance, the thickness of the shell relative to the interior size, and the extreme values and gradients (and thus realization difficulty) of the required permittivity and permeability. Numerical simulations demonstrate the strong scatter reduction performance of the shell and the explicit tradeoffs between performance, shell thickness, and material complexity.

To begin we consider a general 2D radial transformation defined by $r'=f(r)$. Following Ref. 1, the relative electromagnetic permittivity and permeability components can be written in terms of $f(r)$ as[4]

$$\epsilon_r=\mu_r=\frac{f(r)}{rf'(r)},\quad \epsilon_\phi=\mu_\phi=\frac{rf'(r)}{f(r)},\quad \epsilon_z=\mu_z=\frac{f(r)f'(r)}{r}. \tag{1}$$

Because $\epsilon=\mu$ we refer throughout the remainder of the paper to ϵ only with the understanding that the same expressions define μ as well.

To find a cloaking transformation that achieves a given constraint on one of the material components, one simply needs to solve the ordinary differential equation defined by that component and match the boundary conditions of the transformation.[4] Suppose we require that ϵ_ϕ have a constant value k throughout the cloaking shell. The resulting ordinary differential equation for the unknown radial mapping function f is

$$f'=\frac{k}{r}f, \tag{2}$$

which has the general solution

$$f(r)=Ca\left(\frac{r}{a}\right)^k, \tag{3}$$

where C is an integration constant to be determined. Several constants have been inserted into the above expression to leave C dimensionless.

An exact cloak maps the domain $r=[a,b]$ to $r'=[0,b]$ and thus yields the boundary conditions $f(b)=b$ and $f(a)=0$. Equation (3) cannot satisfy this latter condition and thus a strictly ideal 2D cloaking shell cannot have uniform ϵ_ϕ. However, we can apply the approach first described by Ref. 6 and relax this boundary condition by requiring that $f(a)=as$, where s is a dimensionless small value. Analytically, this maps $r=[a,b]$ to $r'=[as,b]$ and results in a cloaking shell that scales the size of the interior of the shell by the factor s and thus "shrinks" the scatterer but does not reduce it to zero size.

[a)]Electronic mail:cummer@ee.duke.edu.
[b)]Electronic mail: tjcui@seu.edu.cn.

Applying the $f(a)=as$ and $f(b)=b$ boundary conditions to Eq. (3) yields

$$C = s, \quad k = 1 - \frac{\ln s}{\ln b/a} \tag{4}$$

(note that $s<1$ and $b>a$ so $k>1$), which results in the cloaking shell material parameters

$$\epsilon_\phi = k, \quad \epsilon_r = k^{-1}, \quad \epsilon_z = s^2 k \left(\frac{r}{a}\right)^{2k-2} \tag{5}$$

(recall that $\mu=\epsilon$). This fully specifies the cloaking shell material parameters in terms of the inner (a) and outer (b) radius of the shell and the desired object scaling factor s.

This solution is entirely unlike the 2D cloaking shell based on a linear mapping in that there are no singularities in any component and the in-plane components are spatially uniform. These properties may make it substantially simpler to realize in practice. Other approaches for removing these singularities[2,4,7] are only valid in the short wavelength limit and their performance is not easy to estimate when this limit is not valid. This solution thus represents a rigorous nonsingular 2D cylindrical electromagnetic cloak that we show below maintains high performance despite the nonsingular parameters.

An aspect of this cloaking shell that provides practical value and insight into its operation is the analytical trade-off it presents between cloaking performance (defined by the object scaling factor s), the thickness of the cloaking shell b/a, and complexity of the required material parameters. Equation (5) shows that increasing k results in-plane permittivity and permeability components farther from unity; ϵ_ϕ gets larger while ϵ_r gets smaller. Increasing k also sharpens the required gradient in ϵ_z near the outer edge of the cloak. Far from unity values and sharp material gradients, and thus larger k values, are more difficult to realize in practice.

Equation (4) then shows explicitly what creates large k values for this cloak. One is small values of the spatial scaling factor s, and the other is values of b/a close to unity, which represents a thin cloaking shell. It is no great surprise that smaller theoretical object compression or a thinner cloaking shell come at the expense of material complexity and thus difficulty in practical realization. This cloaking shell formulation, however, gives explicit expressions that define this trade-off.

Simulations confirm the good performance of this cloaking shell and the straightforward performance-complexity tradeoff it presents. Three separate cases are considered here: $b/a=2$ and $s=0.1$, $b/a=2$ and $s=0.01$, and $b/a=1.5$ and $s=0.1$. Figure 1 plots the radially dependent permittivity and permeability components of the cloaking shell for the three cases.

The commercial package COMSOL MULTIPHYSICS was used to simulate plane wave scattering from objects surrounded by various cloaking shells.[2] The inhomogeneous shell was approximated by a finite element mesh employing approximately 20 layers. Transverse electric polarization was used for both perfect electrical conductor (PEC) and perfect magnetic conductor (PMC) targets, and the resulting plots of

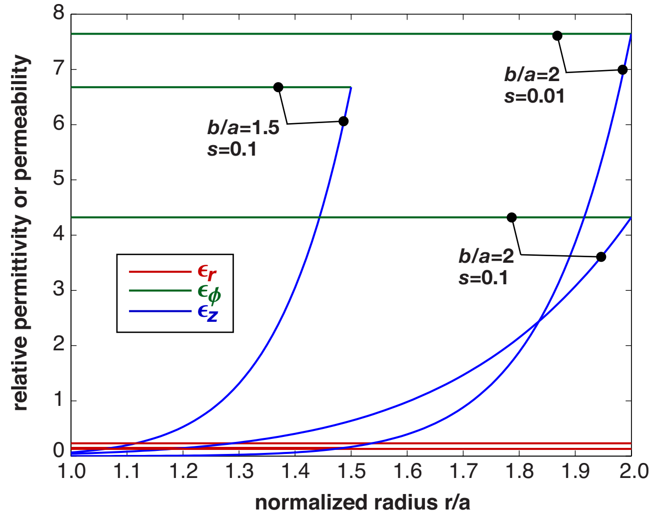

FIG. 1. (Color online) Permittivity and permeability components for the approximate cloaking transformation for three different sets of shell thickness and compression factor.

the real part of the total electric field are shown in Fig. 2 for several cases. The incident wave travels from left to right.

The top row shows scattering from cloaked PEC targets for the three separate cases: $b/a=2$ and $s=0.1$ (left), $b/a=2$ and $s=0.01$ (middle), and $b/a=1.5$ and $s=0.1$ (right). The amount of scattering can be easily judged by the smoothness of the wavefronts. Several expected results are evident. Smaller s yields smaller scattering (compare left and middle), but this comes at the expense of more extreme material parameters (see Fig. 1). A thinner shell can also perform comparably to a thicker shell (compare left and right), but again at the expense of more extreme material parameters.

The results for a PMC target, shown in the bottom row, are similar except for the overall scattering magnitude which will be discussed shortly. Reducing s from 0.1 to 0.01 (left and middle) reduces the scattering, while shrinking the shell thickness b/a while maintaining the same s (left and right) yields essentially identical scattering and thus identical cloaking performance but requires a more challenging set of shell parameters for the thinner cloak.

The significant difference in performance between 2D cloaking shells with PEC and PMC layers on the interior for TE fields has been discussed mathematically in the context of poor convergence of the Bessel function series at the inner edge of the cloak when a PEC layer is present.[8,9] The physical reason for this difference is that a PEC layer at the outer edge of the cloaked region ($r=a$) forces the tangential electric field to zero. For TE incidence, this can be seen in the top panels of Fig. 2 and is inconsistent with the intended spatial transformation of the electric field which should leave the electric field nonzero along this surface. In contrast, a PMC layer at $r=a$ can support a discontinuity of the tangential electric field and thus the mapping is consistent with the boundary conditions. The boundary condition inconsistency for the TE/PEC and equivalent TM/PMC case increases the scattering significantly, although it does not completely degrade the operation of the cloaking shell, as Fig. 2 and prior

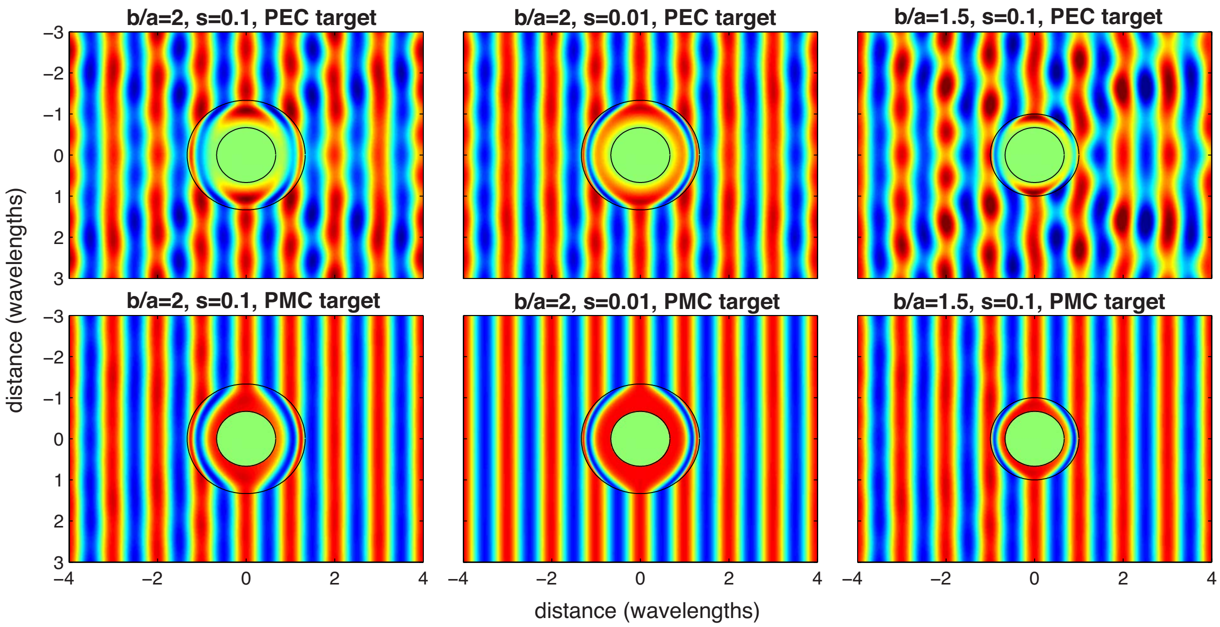

FIG. 2. (Color online) Simulated electric field distributions for three different sets of shell thickness and compression factor for both PEC and PMC shell interiors. The scattering from different interiors is equivalent to scattering from TE and TM polarizations.

work[2,8,9] shows. There is no boundary condition inconsistency for the TE/PMC and equivalent TM/PEC cases, and the cloaking performance is significantly better. This effect can also be explained from the viewpoint of the cloaking shell shrinking the interior object to a small but nonzero size. A thin PEC wire scatters significantly for TE incidence even if the wire is very thin, while a thin PMC wire scatters TE fields much less strongly, as does a thin PEC wire for TM incidence.

It should also be emphasized that even though this cloaking transformation is not strictly exact, its performance in simulation is comparable to the exact linear transformation.[2] We attribute this to the discretization required to simulate an ideal cloak, which renders it no longer ideal. Some similar approximation is likely to be required to construct cloaking shells,[10] and consequently we expect our not quite analytically perfect cloaking shell may perform in practice as well as one based on an ideal formulation, particularly given the relaxed material constraints that our formulation provides.

To conclude, we have derived a rigorous but approximate 2D electromagnetic cloaking shell with nonsingular parameters that make it more physically realizable than those available from analytically exact 2D cloaking shells that contain infinite values. This formulation also yields an explicit analytical tradeoff between the scatter reduction performance of the shell, its thickness relative to the interior size, and the gradients and extreme values of the required material parameters. This trade-off shows rigorously that a thinner or higher performance cloaking shell comes at the expense of one that is more difficult to realize in practice.

T.J.C. acknowledges the supports from the National Basic Research Program (973) of China under Grant No. 2004CB719802 and the National Science Foundation of China.

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