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Material parameters and vector scaling in transformation acoustics

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Abstract. The degree to which the coordinate transformation concept first demonstrated for electromagnetic waves can be applied to other classes of waves remains an open question. In this work, we thoroughly examine the coordinate transformation invariance of acoustic waves. We employ a purely physical argument to show how the acoustic velocity vector must transform differently than the **E** and **H** fields in Maxwell's equations, which explains why acoustic coordinate transformation invariance was not found in some previous analyses. A first principles analysis of the acoustic equations under arbitrary coordinate transformations confirms that the divergence operator is preserved only if velocity transforms in this physically correct way. This analysis also yields closed-form expressions for the bulk modulus and mass density tensor of the material required to realize an arbitrary coordinate transformation on the acoustic fields, which we show are equivalent to forms presented elsewhere. We demonstrate the computation of these material parameters in two specific cases and show that the change in velocity and pressure gradient vectors under a nonorthogonal coordinate transformation is precisely how these vectors must change from purely physical arguments. This analysis confirms that all of the electromagnetic devices and materials that have been conceived using the coordinate transformation approach are also in principle realizable for acoustic

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waves. Together with previous work, this analysis also shows how the curl, divergence and gradient operators maintain form under arbitrary coordinate transformations, opening the door to analyzing other wave systems built on these three vector operators.

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1. Introduction

Pendry *et al* (2006) showed that arbitrary coordinate transformations of Maxwell's equations can be interpreted in terms of an electromagnetic material in the original coordinates with transformed permittivity and permeability values. Consequently, the bending and stretching of electromagnetic fields specified by coordinate transformations can be implemented with electromagnetic materials, enabling unexpected and interesting solutions such as electromagnetic cloaking (Pendry *et al* 2006, Schurig *et al* 2006), and others (Luo *et al* 2008, Rahm *et al* 2008).

The degree to which this coordinate transformation concept can be applied to other classes of waves remains an open question. An analysis by Milton *et al* (2006) indicated that the coordinate transform approach cannot be extended to elastodynamic waves in solids in the fully general case or even for the special case of compressional waves in a fluid, i.e. acoustics. However, a scattering theory analysis has shown that the cloaking solution exists for acoustic waves in fluids in three-dimensions (3D) (Cummer *et al* 2008) and, by analogy with electromagnetics, it has been shown that 2D acoustic waves (Cummer and Schurig 2007) and 3D acoustic waves (Chen and Chan 2007) can be made transformation invariant. The material parameters required to implement acoustic coordinate changes have also been obtained by Greenleaf *et al* (2008).

Some important pieces of physical understanding remain incomplete, though. Demonstrating the invariance through analogy with electromagnetics or by a general analysis of the scalar Helmholtz equation masks some of the physics of the transformation approach, particularly how vectors such as particle velocity and the pressure gradient change under transformation. Our findings reported here are as follows. Through an analysis of how power flow and constant phase surfaces must transform for completely general waves, we show that the velocity vector in acoustics must transform in a different way than the **E** and **H** vectors in electromagnetics. By itself this explains why previous elastodynamic analysis (Milton *et al* 2006), which assumed that the acoustic velocity transforms like **E** and **H**, did not result in acoustic equation transformation invariance.

Then we derive from first principles the conditions on material properties and the scaling of vectors that must be met for the acoustic equations to be coordinate transformation invariant.

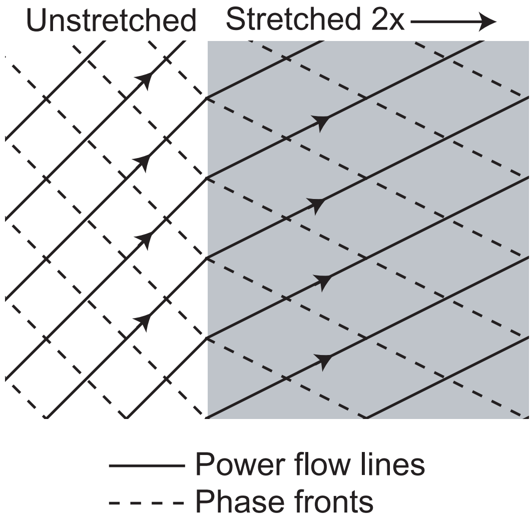

Figure 1. The deformation of power flow lines and phase fronts for a general uniform plane wave after a linear coordinate transformation in one direction. This deformation is the same for all types of waves (electromagnetic, acoustic, etc).

This confirms the material parameter expressions found previously by analogy and also demonstrates how the velocity vector must transform to maintain invariance, which is in agreement with that found from our qualitative argument based on wave physics. Taken together, this analysis and the corresponding derivation for electromagnetics (Pendry *et al* 2006) also show how the complete set of curl, divergence and gradient operators maintain form invariance under arbitrary coordinate transformations, opening the door to analyzing other wave systems built on these three vector operators. Several examples demonstrate the connection between the math and the physics in how the fields transform in orthogonal and nonorthogonal cases.

2. General wave behavior under transformations

Pendry *et al* (2006) noted that conserved vectors in the electromagnetic system, namely the magnetic field **B**, the electric displacement **D** and the Poynting vector **S**, transform in a certain way in order to preserve the form of the Maxwell equations. Physically relevant but nonconserved vectors, however, transform in a different way. For example, in a uniform plane wave the wave vector (or phase front normal) **k** and the Poynting vector **S** are parallel, but after the plane wave has been distorted by a cloaking transformation (Pendry *et al* 2006), these vectors are no longer parallel in the transformed region and they thus do not transform the same way.

To understand this further, we take as a starting point the notion that coordinates transformations stretch nonvector objects, including scalar fields, power flow lines and surfaces of constant phase, as they would be stretched if they were tied to space itself. Consequently, if a region of space containing a uniform plane wave is stretched in one direction, the power flow lines and constant phase surfaces (lines in 2D) distort as shown in figure 1.

Although phase fronts and power flow are universal wave concepts, the vectors involved in different kinds of waves have different relationships to the phase front normal (i.e. the wave

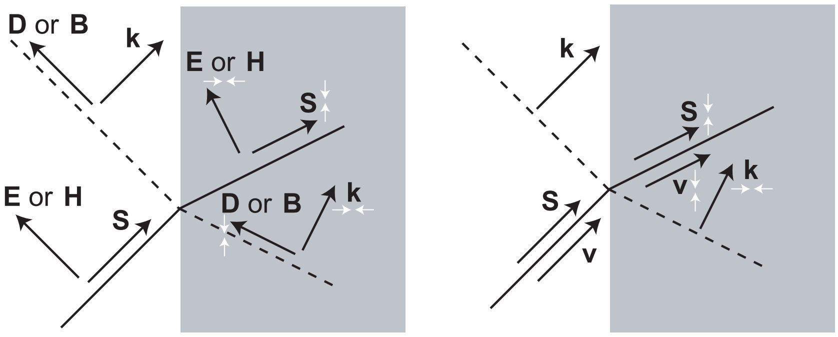

Figure 2. The transformation of vectors in electromagnetic (left) and acoustic or compressional elastodynamic (right). The white converging arrows denote which component of each vector is compressed by the coordinate transformation.

vector) and power flow directions. For example, electromagnetic waves contain **E** and **H** vectors that are perpendicular to the power flow **S**, and **D** and **B** vectors that are perpendicular to the wave vector **k**. How these electromagnetic field vectors must transform in a coordinate transformation is thus illustrated in the left panel of figure 2. The vectors that are conserved, namely **B**, **D** and **S**, must conserve their components parallel to the transformation direction. Thus their components perpendicular to the transformation direction must be compressed (denoted in the figure by the converging white arrows) by the coordinate stretching factor so that they maintain the correct orientation with respect to the power flow lines and phase fronts. The nonconserved vectors, namely **E**, **H** and **k**, must conserve their components perpendicular to the transformation direction. Thus their components parallel to the transformation direction must be compressed by the coordinate stretching factor. Thus the nonconserved vectors **E** and **H** are stretched or compressed in the same direction as the coordinate transformation, while the conserved vectors are stretched or compressed in the direction orthogonal to the coordinate transformation.

The same idea can be applied to compressional elastodynamic (i.e. acoustic) waves, as shown in the right panel of figure 2. Power flow **S** and particle velocity **v** are conserved vectors and therefore must transform so that their components normal to the stretched coordinate direction are altered. Thus, the vector component in acoustic waves, **v**, must transform in a fundamentally different way than the vectors **E** and **H** that describe electromagnetic waves. This immediately explains why the analysis of Milton *et al* (2006) did not reveal the possibilities of acoustic cloaking or, more generally, transformation acoustics. The analysis assumed that the displacement vector **u** (equivalent to **v**) is compressed in the direction of the coordinate transformation, in the manner of **E** and **H**. The first principles analysis below shows that when the velocity or displacement vectors are allowed to transform in the physically correct way, the acoustic field equations are invariant to coordinate transformations, leading to cloaking and all other coordinate transformation-based devices.

3. Direct derivation of transformation acoustics

Recognizing that **v** in an acoustic wave must transform differently than **E** or **H** in an electromagnetic wave, we can show directly that the form of the acoustic equations,

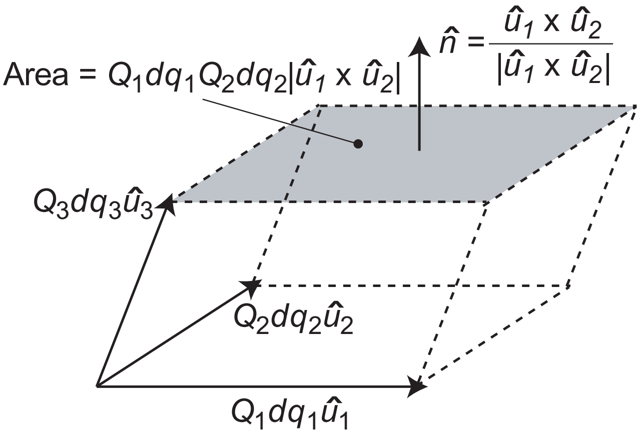

Figure 3. The parallelepiped that defines an infinitesimal volume in the transformed coordinates. The area and unit normal of each face enters in the calculation of the net flux of a vector out of this volume.

namely

$$\nabla p = \mathrm{i}\omega\rho(\bar{r})\rho_0\mathbf{v}, \tag{1}$$

$$\mathrm{i}\omega p = \lambda(\bar{r})\lambda_0\nabla\cdot\mathbf{v}, \tag{2}$$

where $\rho(\bar{r})$ and $\lambda(\bar{r})$ are the normalized density and bulk modulus, respectively, of the medium, is coordinate transform invariant and therefore demonstrate the concept of transformation acoustics in its full form, including expressions for the material parameters and how $\mathbf{v}$ must transform. Following the analysis of the curl operator (Pendry *et al* 2006), we first consider an acoustic wave velocity field $\mathbf{v}$ in a nonorthogonal coordinate system described by coordinates q_1, q_2 and q_3 with unit vectors $\hat{u}_1$, $\hat{u}_2$ and $\hat{u}_3$, respectively. As in Pendry *et al* (2006), let for $i = 1, 2, 3$

$$Q_i^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2. \tag{3}$$

Figure 3 shows what happens when we apply the divergence theorem to an infinitesimal volume in this nonorthogonal coordinate system. Deriving the net outward flux of $\mathbf{v}$ from this volume and setting it equal to the divergence of $\mathbf{v}$ times the infinitesimal volume, it is straightforward to show that

$$(\nabla\cdot\mathbf{v})Q_1Q_2Q_3|\hat{u}_1\cdot(\hat{u}_2\times\hat{u}_3)| = \frac{\partial}{\partial q_1}\left[Q_2Q_3\mathbf{v}\cdot(\hat{u}_2\times\hat{u}_3)\right] + \frac{\partial}{\partial q_2}\left[Q_1Q_3\mathbf{v}\cdot(\hat{u}_1\times\hat{u}_3)\right] + \frac{\partial}{\partial q_3}\left[Q_1Q_2\mathbf{v}\cdot(\hat{u}_1\times\hat{u}_2)\right]. \tag{4}$$

Going forward, let $V_{\mathrm{frac}} = |\hat{u}_1\cdot(\hat{u}_2\times\hat{u}_3)|$ because this is the fraction by which a unit volume is compressed by the coordinate nonorthogonality, and we use the conventional superscript (subscript) notation for contravariant (covariant) vector components. Using

$$\mathbf{v}\cdot(\hat{u}_2\times\hat{u}_3) = v^1\hat{u}_1\cdot(\hat{u}_2\times\hat{u}_3), \tag{5}$$

equation (4) can be rewritten as

$$(\nabla\cdot\mathbf{v})Q_1Q_2Q_3V_{\rm frac}=\frac{\partial}{\partial q_1}\left(Q_2Q_3V_{\rm frac}v^1\right)+\frac{\partial}{\partial q_2}\left(Q_1Q_3V_{\rm frac}v^2\right)+\frac{\partial}{\partial q_3}\left(Q_1Q_2V_{\rm frac}v^3\right). \tag{6}$$

Noting that the divergence in the transformed coordinates is defined by $\nabla_q\cdot\mathbf{v}=\partial v^1/\partial q_1+\partial v^2/\partial q_2+\partial v^3/\partial q_3$, we can write

$$(\nabla\cdot\mathbf{v})Q_1Q_2Q_3V_{\rm frac}=\nabla_q\cdot\left(V_{\rm frac}\bar{\bar{Q}}_{\rm per}[v^1\ \ v^2\ \ v^3]^T\right)=\nabla_q\cdot\tilde{\mathbf{v}}, \tag{7}$$

where

$$\bar{\bar{Q}}_{\rm per}=\begin{bmatrix}Q_2Q_3 & 0 & 0\\ 0 & Q_1Q_3 & 0\\ 0 & 0 & Q_1Q_2\end{bmatrix} \tag{8}$$

and the transformed velocity vector $\tilde{\mathbf{v}}$ is given by

$$\tilde{\mathbf{v}}=V_{\rm frac}\bar{\bar{Q}}_{\rm per}[v^1\ \ v^2\ \ v^3]^T. \tag{9}$$

We use the *per* subscript on the tensor $\bar{\bar{Q}}_{\rm per}$ to denote that the diagonal elements transform each vector component by the product of the coordinate scaling factors *perpendicular* (more generally, not parallel, for the case of nonorthogonal coordinates) to the direction of the vector component. Recall that our qualitative discussion above, summarized in the right panel of figure 2, showed that this is precisely how the velocity vector must transform in a compressional wave in order for transformation acoustics to work. Note that the elements of the column vector $[v^1\ \ v^2\ \ v^3]^T$ are the *contravariant* components of $\mathbf{v}$ in the nonorthogonal coordinate system while the elements of the vector $\tilde{\mathbf{v}}$ are the components in the original orthogonal coordinate system. The examples given in the next section help clarify this distinction. Contrast this to the electromagnetic case (Pendry *et al* 2006) in which the transformed electric field $\tilde{\mathbf{E}}$ is given by

$$\tilde{\mathbf{E}}=[Q_1E_1\ \ Q_2E_2\ \ Q_3E_3]^T. \tag{10}$$

where the scaling factors *parallel* to each component are applied to the *covariant* components of the original electric field in the new coordinate system to yield the transformed field in the original coordinate system. The $\mathbf{H}$ field transforms the same way.

Multiplying (2) (with $\lambda(\bar{r})=1$) by $Q_1Q_2Q_3V_{\rm frac}$ and using (9) results in the equation, in the transformed coordinates,

$$\mathrm{i}\omega p=\lambda(\bar{q})\lambda_0\nabla_q\cdot\tilde{\mathbf{v}}, \tag{11}$$

with

$$\lambda(\bar{q})=(Q_1Q_2Q_3V_{\rm frac})^{-1}. \tag{12}$$

This demonstrates the coordinate transformation invariance of (2) provided the bulk modulus is modified according to (12) and the velocity vector is transformed according to (11). More generally, this also shows how a vector must transform in order for the gradient operator to maintain its basic form.

Now we derive how (1) and therefore the gradient operator transforms under a coordinate change. Using the gradient theorem and integrating ∇p along a short length in the q_1 coordinate direction, we find that

$$\nabla p\cdot Q_1\hat{u}_1=\frac{\partial p}{\partial q_1}=(\nabla_q p)^1. \tag{13}$$

The left-hand side contains the scaled covariant components of ∇p which must be converted to covariant components before it can be equated component-wise to $\nabla_q p$, the gradient in the transformed coordinates. We therefore find that

$$\nabla_q p = \bar{\bar{Q}}_{\mathrm{par}} \bar{\bar{h}}^{-1} (\nabla p), \tag{14}$$

where $\bar{\bar{Q}}_{\mathrm{par}}$ is the diagonal tensor containing coordinate scaling factors *parallel* to the direction of the vector component, or

$$\bar{\bar{Q}}_{\mathrm{par}} = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix} \tag{15}$$

and

$$\bar{\bar{h}}^{-1} = \begin{bmatrix} \hat{u}_1 \cdot \hat{u}_1 & \hat{u}_1 \cdot \hat{u}_2 & \hat{u}_1 \cdot \hat{u}_3 \\ \hat{u}_2 \cdot \hat{u}_1 & \hat{u}_2 \cdot \hat{u}_2 & \hat{u}_2 \cdot \hat{u}_3 \\ \hat{u}_3 \cdot \hat{u}_1 & \hat{u}_3 \cdot \hat{u}_2 & \hat{u}_3 \cdot \hat{u}_3 \end{bmatrix}. \tag{16}$$

Note that this $\bar{\bar{h}}^{-1}$ is the same as $\bar{\bar{g}}^{-1}$ defined by Pendry *et al* (2006). We rename this tensor because we will use $\bar{\bar{g}}$ later to denote the metric tensor which is not quite the same as this $\bar{\bar{h}}$.

Finally, multiplying (1) (with $\rho(\bar{r}) = 1$) by $\bar{\bar{Q}}_{\mathrm{par}} \bar{\bar{h}}^{-1}$, we find

$$\nabla_q p = \mathrm{i}\omega \bar{\bar{Q}}_{\mathrm{par}} \bar{\bar{h}}^{-1} \rho_0 \mathbf{v} = \mathrm{i}\omega \bar{\bar{Q}}_{\mathrm{par}} \bar{\bar{h}}^{-1} \bar{\bar{Q}}_{\mathrm{per}}^{-1} V_{\mathrm{frac}}^{-1} \rho_0 \tilde{\mathbf{v}} \tag{17}$$

leaving us with the equivalent of (1) in fully transformed coordinates

$$\nabla_q p = \mathrm{i}\omega \bar{\bar{\rho}} \rho_0 \tilde{\mathbf{v}} \tag{18}$$

with

$$\bar{\bar{\rho}} = \bar{\bar{Q}}_{\mathrm{par}} \bar{\bar{h}}^{-1} \bar{\bar{Q}}_{\mathrm{per}}^{-1} V_{\mathrm{frac}}^{-1}. \tag{19}$$

Equations (11) and (18) show that the acoustic equations are fully transformation invariant with the modified material parameters in (12) and (19). We further show below that these expressions are equivalent to those shown by Chen and Chan (2007) purely by analogy with electromagnetics through the electric conductivity equation (Greenleaf *et al* 2003) and those derived by Greenleaf *et al* (2008) for the general scalar Helmholtz equation. Consequently cloaking shells, concentrators and other devices that have been designed theoretically for electromagnetics can also be realized for acoustics provided that the bulk modulus and anisotropic effective mass density tensor can be realized in practice as specified by (12) and (19). Importantly, this first principles derivation shows explicitly in (9) how the acoustic velocity vector must transform under coordinate changes, which, as noted above, is different from how the **E** and **H** fields transform in electromagnetics. The scalar pressure is, however, not changed by the coordinate transformation and thus, like phase fronts and power flow lines, is simply deformed by any coordinate transformation.

4. Equivalent forms and special cases

The material parameters required to realize a particular coordinate transformation in (12) and (19) can also be expressed in terms of the metric tensor. Noting that $\bar{\bar{Q}}_{\mathrm{per}} = Q_1 Q_2 Q_3 \bar{\bar{Q}}_{\mathrm{par}}^{-1}$, (19) can be rewritten as the inverse mass density tensor as

$$\bar{\bar{\rho}}^{-1} = (Q_1 Q_2 Q_3 V_{\text{frac}})\bar{\bar{Q}}_{\text{par}}^{-1}\bar{\bar{h}}\bar{\bar{Q}}_{\text{par}}^{-1}. \tag{20}$$

It is straightforward to show that the metric tensor g^{ij} can be expressed as

$$g^{ij} = \bar{\bar{g}} = \bar{\bar{Q}}_{\text{par}}^{-1}\bar{\bar{h}}\bar{\bar{Q}}_{\text{par}}^{-1}, \tag{21}$$

and, recognizing that $\det(\bar{\bar{Q}}_{\text{par}}) = Q_1 Q_2 Q_3$ and $\det(\bar{\bar{h}}^{-1}) = V_{\text{frac}}^2$, we also have

$$\det(g^{ij}) = \left|g^{ij}\right| = (Q_1 Q_2 Q_3 V_{\text{frac}})^{-2}. \tag{22}$$

Therefore, the transformed bulk modulus and mass density tensors can be expressed as

$$\bar{\bar{\rho}}^{-1} = \frac{\bar{\bar{g}}}{\sqrt{\left|g^{ij}\right|}}, \tag{23}$$

$$\lambda = \sqrt{\left|g^{ij}\right|}, \tag{24}$$

which is equivalent to that reported in Greenleaf *et al* (2008). Note that the inverse mass density tensor here is identical to the permittivity and permeability tensors for electromagnetics (Schurig *et al* 2006) under the same coordinate transformation.

The special equivalence in 2D between acoustics and transverse electric (TE) and transverse magnetic (TM) polarization electromagnetics (Cummer and Schurig 2007) is a straightforward consequence of the above result. If $Q_3 = 1$ and $\hat{u}_1 \cdot \hat{u}_3 = \hat{u}_2 \cdot \hat{u}_3 = 0$, and recalling that the inverse mass density tensor $\bar{\bar{\rho}}^{-1}$ for a given transformation in acoustics is the same as the permittivity $\bar{\bar{\epsilon}}$ and the permeability $\bar{\bar{\mu}}$ for the same transformation in electromagnetics, then

$$\bar{\bar{\rho}}^{-1} = \bar{\bar{\epsilon}} = \bar{\bar{\mu}} = Q_1 Q_2 V_{\text{frac}} \begin{bmatrix} \dfrac{\hat{u}_1 \cdot \hat{u}_1}{Q_1^2} & \dfrac{\hat{u}_1 \cdot \hat{u}_2}{Q_1 Q_2} & 0 \\ \dfrac{\hat{u}_2 \cdot \hat{u}_1}{Q_1 Q_2} & \dfrac{\hat{u}_2 \cdot \hat{u}_2}{Q_2^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{25}$$

and

$$\lambda^{-1} = Q_1 Q_2 V_{\text{frac}}. \tag{26}$$

Thus, for TE fields, $\lambda^{-1} = \epsilon_z$. And for the special case of an orthogonal coordinate transformation, all off diagonal elements are zero and the two upper diagonal elements simply invert when the matrix is inverted to give the density $\bar{\bar{\rho}}$, giving

$$\mu_{xx} = \rho_{yy} = Q_1/Q_2, \tag{27}$$

$$\mu_{yy} = \rho_{xx} = Q_2/Q_1. \tag{28}$$

Thus the equivalence of transposed diagonal μ and ρ components and of λ^{-1} and ϵ_z found in (Cummer and Schurig 2007) occurs in 2D for orthogonal transformations, such as the cloaking transformation. The equivalence for TM fields is similar.

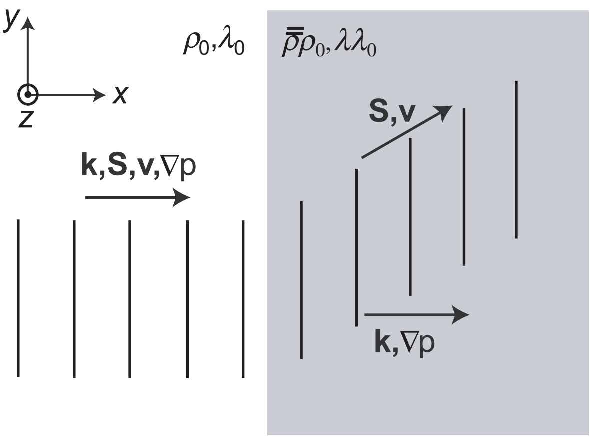

Figure 4. Schematic of an acoustic beam normally incident on a beam shifting material defined by an x dependent shift of the y-coordinate. The relationship between the **v** and ∇p vectors, the beam direction and the wave vector are shown for both materials.

5. Orthogonal and nonorthogonal examples

We now apply the theory discussed above for two specific cases to illustrate the theory. Firstly, consider the spherical cloaking transformation (Pendry *et al* 2006) specified by $r' = a + r(b-a)/b$, where a and b are constants and $b > a$. This coordinate transformation is orthogonal and thus $\bar{\bar{h}} = I$ and $V_{\text{frac}} = 1$, which simplify things considerably. The Q_i length scaling factors are straightforward to calculate provided one realizes that the azimuthal and polar angles are not lengths as in cartesian coordinates and (3) must be modified slightly. The Q_i are defined by the ratio of infinitesimal lengths in the transformed and untransformed coordinates, and thus

$$Q_r = \frac{\mathrm{d}r}{\mathrm{d}r'} = \frac{b}{b-a}, \qquad Q_\phi = \frac{r\,\mathrm{d}\phi}{r'\,\mathrm{d}\phi'} = \frac{b}{b-a}\frac{r'-a}{r'}, \qquad Q_\theta = \frac{r\sin(\theta)\,\mathrm{d}\theta}{r'\sin(\theta')\,\mathrm{d}\theta'} = Q_\phi, \tag{29}$$

which from (19) and (12) yield a diagonal mass density tensor and complete material properties of

$$\rho_r = \frac{b-a}{b}\left(\frac{r'}{r'-a}\right)^2, \qquad \rho_\phi = \rho_\theta = \frac{b-a}{b}, \qquad \lambda = \left(\frac{b-a}{b}\right)^3\left(\frac{r'}{r'-a}\right)^2, \tag{30}$$

in agreement with the parameters found previously through other approaches (Chen and Chan 2007, Cummer *et al* 2008, Greenleaf *et al* 2008).

Next, we consider the nonorthogonal coordinate transformation of the beam shifter described by Rahm *et al* (2008) and summarized in figure 4. The coordinate transformation from cartesian coordinates (x, y, z) to the new coordinates (x', y', z') is

$$x' = x, \quad y' = y + ax, \quad z' = z, \tag{31}$$

which yields the scaling factors

$$Q_1 = \sqrt{1+a^2}, \quad Q_2 = 1, \quad Q_3 = 1. \tag{32}$$

The unit basis vectors in the new coordinate system are (Rahm *et al* 2008)

$$\hat{x}' = \frac{\hat{x} - a\hat{y}}{\sqrt{1+a^2}}, \quad \hat{y}' = \hat{y}, \quad \hat{z}' = \hat{z}. \tag{33}$$

which give $V_{\rm frac} = (1+a^2)^{-1/2}$ and

$$\bar{\bar{h}}^{-1} = \begin{bmatrix} 1 & -\frac{a}{\sqrt{1+a^2}} & 0 \\ -\frac{a}{\sqrt{1+a^2}} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{34}$$

Together with (19) and (12) these yield the material parameters for the acoustic beam shifter

$$\bar{\bar{\rho}} = \begin{bmatrix} \frac{1}{1+a^2} & -a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{35}$$

and

$$\lambda = 1. \tag{36}$$

It is also useful to examine the directions of physically relevant vectors in the transformed medium. In acoustic waves, power flow is parallel to **v** and thus, in the beam shifter material, **v** should point in the direction of the beam, which is $\hat{x} + a\hat{y}$ in the original cartesian basis. To confirm this using (9), we first compute the contravariant components of the original velocity $v_0\hat{x}$ in the transformed basis (33), which are

$$v^1 = \frac{v_0}{\sqrt{1+a^2}}, \quad v^2 = av_0, \quad v^3 = 0. \tag{37}$$

It is easy to confirm that $v^1\hat{x}' + v^2\hat{y}' + v^3\hat{z}' = v_0\hat{x}$. Each of these components is then scaled as in (9) and when expressed in the original basis, we have

$$\tilde{v} = V_{\rm frac}(Q_2Q_3v^1\hat{x} + Q_1Q_3v^2\hat{y}) = v_0(\hat{x} + a\hat{y}). \tag{38}$$

This is precisely the velocity required in the new medium to steer the beam in the new direction but also conserve the velocity component normal to the interface with free space.

The wave vector $k_0\hat{x}$ in the transformed medium also changes as it should according to basic wave physics. As described in section 2 above, the wave vector is not conserved and thus transforms like **E** or **H** in electromagnetics, as defined by (10). First, we compute the covariant components of the original wave vector in the transformed basis, which are

$$k_1 = \frac{k_0}{\sqrt{1+a^2}}, \quad k_2 = 0, \quad k_3 = 0. \tag{39}$$

When scaled and expressed in the original basis, we have

$$\tilde{k} = Q_1k_1\hat{x} = k_0\hat{x}, \tag{40}$$

which again is precisely what is needed to conserve the component of **k** transverse to the interface with free space and maintain the phase front normal in the $\hat{x}$-direction as shown in figure 4.

Note that the above procedure also shows how one can compute the directions of electromagnetic fields in media derived using a transformation optics approach. **E** and **H** are nonconserved vectors and thus transform like **k** above, while **B**, **D**, **S** and **J** (current density) are all conserved and thus transform exactly as **v** above. Vectors in other wave systems will also transform in these ways, again depending on whether they are conserved or nonconserved vectors.

6. Conclusions

Through a purely physical argument based on how power flow and phase fronts change under coordinate transformations for all types of waves, we showed how vectors in electromagnetic (**E** and **H**) and acoustic waves (**v**) must transform differently under coordinate transformations. This explains in simple terms why a thorough analysis of the coordinate transformation invariance of elastodynamics (Milton *et al* 2006) in which **v** was assumed to transform like **E** did not show invariance even for the limited case of acoustics. A first principles analysis of the acoustic equations under arbitrary coordinate transformations confirms that the divergence operator is preserved only if velocity transforms in this physically correct way. This analysis also shows directly that the acoustic equations are coordinate transformation invariant, as has been demonstrated by analogy (Chen and Chan 2007) with the electric conductivity equations (Greenleaf *et al* 2003) and by analysis of the scalar Helmholtz equation (Greenleaf *et al* 2008). We show that these different expressions for the bulk modulus and mass density tensor needed to realize an arbitrary deformation of the acoustic velocity and pressure fields are fundamentally equivalent. We demonstrate the computation of these material parameters in two specific cases and show that the change in velocity and pressure gradient vectors under a nonorthogonal coordinate transformation is precisely how these vectors must change from purely physical arguments.

This further confirms that all of the electromagnetic devices and materials that have been conceived using the coordinate transformation approach (Luo *et al* 2008, Rahm *et al* 2008), are also in principle realizable for acoustic waves. Methods for creating composite materials to realize the required anisotropic acoustic materials have already been described (Cheng *et al* 2008, Torrent and Sanchez-Dehesa 2008a, Torrent and Sanchez-Dehesa 2008b). Together, this acoustic analysis and the corresponding electromagnetic analysis (Pendry *et al* 2006) show how the curl, divergence and gradient operators maintain form under arbitrary coordinate transformations, opening the door to analyzing other wave systems built on these three vector operators.

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