

Optical Design of Reflectionless Complex Media by Finite Embedded Coordinate Transformations

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Transformation optics offers an unconventional approach to the control of electromagnetic fields. The transformation optical structures proposed to date, such as electromagnetic “invisibility” cloaks and concentrators, are inherently reflectionless and leave the transmitted wave undisturbed. Here, we expand the class of transformation optical structures by introducing finite, embedded coordinate transformations, which allow the electromagnetic waves to be steered or focused. We apply the method to the design of several devices, including a parallel beam shifter and a beam splitter, both of which are reflectionless and exhibit unusual electromagnetic behavior as confirmed by 2D full-wave simulations.

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Recently, the technique of transformation optics has emerged as a means of designing complex media that can bring about unprecedented control of electromagnetic fields. A transformation optical structure is designed by first applying a form-invariant coordinate transform to Maxwell’s equations, in which part of free space is distorted in some manner. The coordinate transformation is then applied to the permittivity and permeability tensors to yield the specification for a complex medium with desired functionality [1]. Although such media would be very difficult to synthesize using conventional materials, artificially structured metamaterials offer a practical approach for the realization of transformation optical designs, given their ability to support a much wider and controllable range of electric and magnetic responses. A metamaterial version of an invisibility cloak, designed using the transformation optical technique, was constructed and successfully used to confirm the theoretically predicted cloaking mechanism [2]. This work has generated widespread interest specifically in the prospects of electromagnetic cloaking, one of the more dramatic electromagnetic phenomena that can be enabled using transformation optical structures [3–15].

All of the transformation optical designs reported in the literature thus far have in common that the electromagnetic properties of the incident waves are altered exclusively within a restricted region of the transformation optical device. For these transformations, the alteration of the field structure cannot be transferred to another medium or to free space and thus remains a local phenomenon. That is, the structures designed by transformation optics are inherently invisible, whether they are designed to be cloaks or other devices. Optical elements, such as lenses, that would either focus or steer electromagnetic waves would appear to be precluded were the transformation optical approach strictly followed.

In this Letter, we expand the class of transformation optical structures by the use of finite, embedded coordinate

transformations. Embedded transformations add a significant amount of flexibility to the transformation design of complex materials, enabling the transfer of field manipulations from the transformation optical medium to a second medium. In this way, lenses and other optical elements that control and guide light can be designed, although with unconventional properties. Like transformation optical devices, the finite-embedded transform structures can also be reflectionless when certain criteria are met.

The mathematical formalism used for the calculation of the complex material properties is similar to the one reported in [6,16]. Throughout this Letter, we use the same terminology as introduced in these references.

For a given coordinate transformation $x^{\alpha'}(x^\alpha) = A_\alpha^{\alpha'} x^\alpha$ ($A_\alpha^{\alpha'}$: Jacobi matrix, $\alpha = 1 \dots 3$), the electric permittivity $\epsilon^{i'j'}$ and the magnetic permeability $\mu^{i'j'}$ of the resulting material can be calculated by

$$\epsilon^{i'j'} = [\det(A_i^{i'})]^{-1} A_i^{i'} A_j^{j'} \epsilon^{ij} \quad (1)$$

$$\mu^{i'j'} = [\det(A_i^{i'})]^{-1} A_i^{i'} A_j^{j'} \mu^{ij} \quad (2)$$

where $\det(A_i^{i'})$ denotes the determinant of the Jacobi matrix. For all the transformations presented here, the mathematical starting point is 3-dimensional Euclidian space expressed in Cartesian coordinates with isotropic permittivities and permeabilities $\epsilon^{ij} = \epsilon_0 \delta^{ij}$ and $\mu^{ij} = \mu_0 \delta^{ij}$ (δ^{ij} : Kronecker delta).

A possible coordinate transformation for the design of a parallel beam shifter and a beam splitter consisting of a slab with thickness $2d$ and height $2h$ can be expressed by

$$x'(x, y, z) = x \quad (3)$$

$$y'(x, y, z) = \Theta(h_1 - |y|)[y + ak_l(x, y)] \\ + \Theta(|y| - h_1)[y + \gamma(y)k_l(x, y)(y - s_2(y)h)] \quad (4)$$

$$z'(x, y, z) = z \quad (5)$$

with

$$\Theta: \xi \rightarrow \Theta(\xi) := \begin{cases} 1 & \xi > 0 \\ 1/2 & \xi = 0 \\ 0 & \xi < 0 \end{cases} \quad (6)$$

$$k_l: (\eta, \kappa) \rightarrow k_l(\eta, \kappa) := s_p(\kappa)(\eta + d)^l \quad (7)$$

$$s_p: \xi \rightarrow s_p(\xi) := \begin{cases} 1 & p = 1 \\ +1 & \xi \geq 0 \\ -1 & \xi < 0 \end{cases} \quad p = 2 \quad (8)$$

$$\gamma: \vartheta \rightarrow \gamma(\vartheta) := \frac{a}{s_2(\vartheta)(h_1 - h)} \quad (9)$$

where $2h_1$ is the maximum allowed width of the incoming beam, a determines the shift amount, and $l = 1 \dots n$ is the order of the nonlinearity of the transformation.

The transformation equations are defined for $(|x| \leq d)$, $(|y| \leq h)$ and $(|z| < \infty)$. For the case $p = 1$, Eqs. (3)–(5) describe a parallel beam shifter whereas for $p = 2$ the equations refer to a beam splitter. The Jacobi matrix of the transformation and its determinant are

$$A_i^{j'} = \begin{pmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (10)$$

$$\det(A_i^{j'}) = a_{22} \quad (11)$$

with

$$a_{21} = \Theta(h_1 - |f'_1|)[lak'_{l-1}] \\ + \Theta(|f'_2| - h_1)\{l\gamma'k'_{l-1}[f'_2 - s_2(y')h]\} \quad (12)$$

$$a_{22} = \Theta(h_1 - |f'_1|) + \Theta(|f'_2| - h_1)[1 + \gamma'k'_l] \quad (13)$$

where

$$f'_1: (x', y') \rightarrow f(x', y') := y' - ak'_l \quad (14)$$

$$f'_2: (x', y') \rightarrow f(x', y') := \frac{y' + \gamma'k'_l s_2(y')h}{1 + \gamma'k'_l} \quad (15)$$

$k'_l := k_l(x', y')$ and $\gamma' := \gamma(y')$.

By Eqs. (1) and (2), it is straightforward to calculate the tensors of the transformed relative electric permittivity $\epsilon_r = \epsilon/\epsilon_0$ and the relative magnetic permeability $\mu_r = \mu/\mu_0$, which in the material representation are obtained as

$$\epsilon_r^{ij} = \mu_r^{ij} = \frac{1}{a_{22}} g^{ij} \quad (16)$$

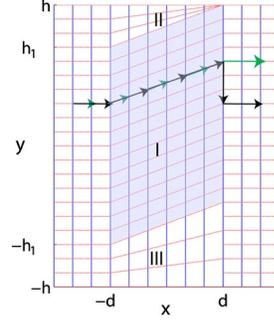


FIG. 1 (color online). Linear spatial coordinate transformation of a parallel beam shifter.

where

$$g^{ij} = \begin{pmatrix} 1 & a_{21} & 0 \\ a_{21} & a_{21}^2 + a_{22}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (17)$$

is the metric tensor of the coordinate transformation. At this point, it should be mentioned that only the domain with $\Theta(h_1 - |f'_1|) \equiv 1$ has to be considered in the material implementation which simplifies the mathematical expressions.

A linear coordinate transformation ($l = 1$) for the design of a parallel beam shifter is illustrated in Fig. 1. The space within the gray shaded region I [$\Theta(h_1 - |f'_1|) \equiv 1$] is tilted at an angle $\phi = \arctan(a)$ with respect to free space, which inherently results in a compression of space in region II and a dilution of space in region III. Considering the boundary between region I and free space, the coordinate systems are seen to be discontinuous at the interface $x = d$. Note that if the transformation was interpreted as discontinuous, the boundary transition between the free space and transformed regions would need to be taken into account. For example, the transformation of the y coordinate at the transition region I and free space would be

$$y'(x, y, z) = \Theta(d - x)[y + ak_l(x, y)] + \Theta(x - d)y \quad (18)$$

so that

$$a_{21}|_{(x=d)} = \Theta(d - x)lak_{l-1}(x, y) - \delta(x - d)ak_l(x, y). \quad (19)$$

Since the y coordinate lines in Fig. 1 represent the direction of the power flow for a beam propagating from $x = -\infty$ to ∞ in the ray approximation, the inclusion of the delta distribution in the material parameters would result in the trajectory indicated by the black arrows. The incoming wave would be shifted in the y direction and abruptly be forced back on its old path at the boundary at $x = d$, thereby rendering the entire beam shifting section invisible to an observer. However, as can be seen from Eq. (4), the boundary is not included in the calculation of the material properties for the beam shifter and beam splitter. The coordinate transformation is carried out locally for the

transformation optical medium and then embedded into free space which results in the trajectory indicated by the green arrows. The beam is shifted in the y direction and maintains its lateral shift after exiting the transformation optical medium. This method is similar to the "embedded Green function" approach in the calculation of electron transport through interfaces [17] so that we call it an "embedded coordinate transformation."

One of the interesting properties of transformation optical devices is that they are inherently reflectionless. It is not obvious that this property should hold for the embedded transformation. By heuristic means, we found as a necessary—and in our investigated cases also sufficient—topological condition that the metric in and normal to the interface between the transformation optical medium and the nontransformed medium (in this case free space) must be continuous to the surrounding space. In order to express this criterion in mathematical terms, we consider a local coordinate system with origin at $x = d$, the canonical, normalized basis vectors $\{\tilde{e}_i\}$, and the metric tensor of euclidian space $g_{ij} = \delta_{ij}$. The basis vectors $\{\tilde{e}_i\}$ are chosen to be parallel and normal to the interface between the beam shifter section and free space. The metric tensor $g_{i'j'}$ of the coordinate transformation (3)–(5) can be readily obtained by applying $g^{\alpha\beta}g_{\beta\gamma} = \delta_\gamma^\alpha$ to (17) as

$$g_{i'j'} = \begin{pmatrix} 1 + a^2 & -a & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \langle \tilde{e}_{i'} | \tilde{e}_{j'} \rangle \quad (20)$$

where $\langle | \rangle$ denotes the scalar product. The basis vectors $\{\tilde{e}_i\}$ can be expressed in terms of the transformed coordinate system by use of the relation $\tilde{e}_i = A_i^{i'} \tilde{e}_{i'}$ and (10) as

$$\tilde{e}_1 = \tilde{e}_{1'} + a\tilde{e}_{2'}, \quad \tilde{e}_2 = \tilde{e}_{2'}, \quad \tilde{e}_3 = \tilde{e}_{3'}. \quad (21)$$

It's easy to see by combining (20) and (21) that the norm $|\tilde{e}_i| = \sqrt{\langle \tilde{e}_i | \tilde{e}_i \rangle}$ of the basis vectors $\{\tilde{e}_i\}$ in the transformed coordinate system equals their norm in the untransformed space, and thus the topological condition is fulfilled for the linear beam shifter transformation.

As a second, more sophisticated example, a beam splitter is presented for the case of a nonlinear transformation of second order. The material properties are described by Eq. (16) with ($p = 2$) and ($l = 2$). This specific coordinate transformation is illustrated in Fig. 2. The underlying metric describes the gradual opening of a wedge-shaped slit in the y direction. The metric in the x direction is not affected by the transformation. Similar to the parallel beam shifter, the beam splitter obeys the topological condition in order to operate without reflection.

To confirm our findings, 2D full-wave simulations were carried out to explore the electromagnetic behavior of waves impinging on a beam shifter and a beam divider, respectively. The calculation domain was bounded by perfectly matched layers. The electric field of the plane waves was chosen to be perpendicular to the x - y plane.

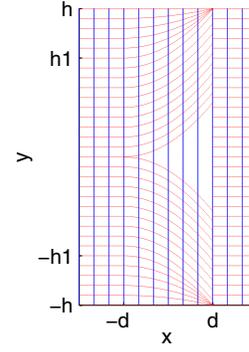


FIG. 2 (color online). Nonlinear spatial coordinate transformation of second order for a beam divider.

Figure 3 depicts the spatial distribution of the real part of the transverse-electric phasor (color map) and the direction of the power flow (gray lines) of propagating waves across a parallel beam shifter at perpendicular [Figs. 3(a)–3(d)] and oblique incidence [Figs. 3(e) and 3(f)]. The curvature of the incoming wave fronts was freely chosen to be plane (a–b) or convergent (c–f). As can be seen from Figs. 3(a) and 3(b), the beam shifter translates the incoming plane wave in the y direction perpendicular to the propagation x direction without altering the angle of the wave fronts. In contrast, the direction of the power flow changes by an angle $\phi = \arctan(a)$ (a : shift parameter) with respect to the power flow of the incoming plane wave. After propagation through the complex transformation optical medium, the wave fronts and the power flow possess the same direction as the incoming beam; however, the position of the wave is offset in the y direction. The shift parameter a was arbitrarily chosen to be 1.8 [Fig. 3(a)] and -1.8 [Fig. 3(b)].

A similar behavior can be observed for waves with wave fronts of arbitrary curvature, as, for example, for convergent waves [Figs. 3(c)–3(f)]. In this case, the focus of the beam can be shifted within a plane parallel to the y -axis by variation of the shift parameter a . As is obvious from Figs. 3(e) and 3(f), the same behavior applies for incoming waves at oblique incidence. The beam solely experiences a translation in the y direction whereas the x -position of the focus remains unchanged. In all cases, the realized trans-

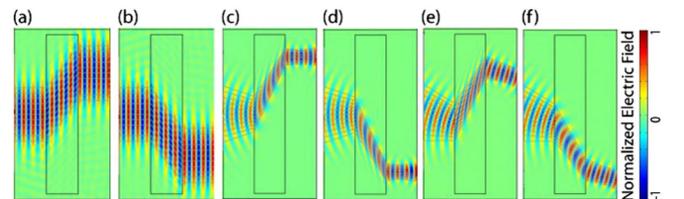


FIG. 3 (color online). Electric field distribution (color map) and power flow lines (gray) of a parallel beam shifter for diffracting plane waves with shift parameters (a) $a = 1.8$, (b) $a = -1.8$, for a convergent beam under perpendicular incidence with (c) $a = 2$, (d) $a = -2$ and for oblique incidence with (e) $a = 2$ and (f) $a = -1.2$.

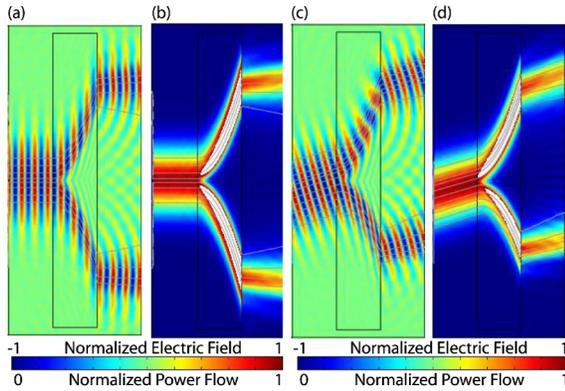


FIG. 4 (color online). Electric field distribution (color map), power flow lines (gray) [(a) and (c)] and power flow [(b) and (d)] of a beam splitter for diffracting plane waves with shift parameters (a)–(b) $|a| = 15$ for perpendicular incident waves and (c)–(d) $|a| = 12$ for oblique incidence. The power flow color maps are saturated for the sake of contrast enhancement.

formation optical parallel beam shifter proves to operate without reflection confirming our metric criterion used for the design. The presented parallel beam shifter could play a crucial role in connection with tunable, reconfigurable metamaterials as it would allow scanning of a beam focus along a flat surface without changing the plane of the focus and without introducing a beam tilt or aberrations. These properties become even more significant for applications where short working distances are used between scanner and object.

Similar simulations were carried out for the transformation optical beam splitter. Figure 4(a) shows the electric field distribution and the power flow lines for waves at perpendicular incidence. The beam splitter shifts one half of the wave in the $(+y)$ direction and the second half in the $(-y)$ direction, thus splitting the wave at the midpoint. The split waves are not perfectly parallel at the exit plane of the device due to diffraction of the incoming wave of finite width. As can be seen, there exists a small fraction of scattered fields within the split region which can be explained in terms of diffraction and scattering which is out of the scope of this Letter. The beam splitter was found to operate without reflection in agreement with the metric criterion.

Figure 4(b) displays the normalized power flow inside and outside the device. In order to enhance the contrast in the visualization of the power distribution at the beam splitter output, the color scale is saturated inside the beam splitter medium. As obvious from the transformation (Fig. 2), the power density inside the transformation optical medium is expected to be higher than outside the material, which is indicated by the density of the grid lines in Fig. 2 and confirmed by the simulations. The power flow density abruptly decreases at the output facet of the beam splitter. By integration of the power density inside the gap region between the beams and the power density inside either the upper or lower arm of the split beams, a power ratio of 10:1

was calculated. The scattered waves in the gap carried about 4% of the total power. A similar, reflectionless splitting behavior could be observed for waves at oblique incidence as illustrated in Figs. 4(c) and 4(d).

In conclusion, the concept of embedded coordinate transformations significantly expands the idea of the transformation optical design of metamaterials to a whole new class of optical devices which are not inherently invisible to an external observer and allow to transfer the electromagnetic field manipulations from the transformation optical medium to another medium. In addition, a heuristic topological design criterion was described in order to engineer the transformation optical device perfectly reflectionless. The new design method was illustrated at the examples of a parallel beam shifter and splitter. Both devices showed an extraordinary electromagnetic behavior which is not achievable by conventional materials. Both examples clearly state the significance of embedded coordinate transformations for the design of new electromagnetic elements with tunable, unconventional optical properties.

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