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One path to acoustic cloaking

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Abstract. A complete analysis of coordinate transformations in elastic media by Milton *et al* has shown that, in general, the equations of motion are not form invariant and thus do not admit transformation-type solutions of the type discovered by Pendry *et al* for electromagnetics. However, in a two-dimensional (2D) geometry, the acoustic equations in a fluid are identical in form to the single polarization Maxwell equations via a variable exchange that also preserves boundary conditions. We confirm the existence of transformation-type solutions for the 2D acoustic equations with anisotropic mass via time harmonic simulations of acoustic cloaking. We discuss the possibilities of experimentally demonstrating acoustic cloaking and analyse why this special equivalence of acoustics and electromagnetics occurs only in 2D.

The transformation-based solutions to the Maxwell equations reported by Pendry *et al* [1] yield a general method for rendering arbitrarily sized and shaped objects electromagnetically invisible. Other approaches for almost eliminating electromagnetic scattering in all directions from certain classes of objects of limited size have been described [2]–[4], and Leonhardt [5, 6] has given a recipe that can cloak an object in the short wavelength limit that thus applies to different classes of waves, including electromagnetic and acoustic. Similar transformation-based approaches have been analysed for the related problem of electrical impedance tomography [7, 8]. The solution found by Pendry *et al* stands out for its theoretically perfect performance regardless of wavelength and object size and shape.

It is undoubtedly of interest whether the transformation-based electromagnetic solutions, which were simulated in [9] and experimentally confirmed in [10], can be extended to waves in other systems. From one perspective, it seems that the answer could be yes. The concepts of wavevector, wave impedance, and power flow are universal, and the manner in which permittivity

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and permeability control these in electromagnetics is closely analogous to that by material properties in other systems. On the other hand, the Maxwell equations possess a special symmetry that is an important element of the equivalence between coordinate transformations and material properties for electromagnetic waves. The field equations that describe waves in most other systems do not have this same symmetry.

Milton *et al* [11] analysed the full equations of motion for a general elastic medium under coordinate transformations and found that, in general, the form of the equations of motion are not invariant to transformations. The possibility of engineering artificial elastic media that obeyed these modified equations of motion was discussed, but no simple scheme for realizing a transformation-based application, such as acoustic cloaking, was reported. There is at least one special case, however, in which the equivalence between electromagnetics and elastodynamics is complete. Moreover, this case is practically useful and can potentially be demonstrated experimentally. It has long been known that in two dimensions (2D) acoustics and electromagnetics in isotropic media are exactly equivalent [e.g., 12]. We show below that this isomorphism holds for anistropic media as well. For an inviscid fluid with zero shear modulus, the linearized equations of state for small amplitude perturbations from conservation of momentum, conservation of mass, and linear relationship between pressure and density are [13]

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p, \tag{1}$$

$$\frac{\partial p}{\partial t} = -\lambda \nabla \cdot \mathbf{v}, \tag{2}$$

where p is scalar pressure, $\mathbf{v}$ is vector fluid velocity, ρ_0 is the unperturbed fluid mass density, and λ is the fluid bulk modulus. This set of equations admits the usual compressional wave solutions in which fluid motion is parallel to the wavevector. Milton *et al* [11] showed that any coordinate transformation on an elastic medium produces an anisotropic mass represented by a second rank tensor. In cylindrical coordinates with z invariance, and letting the mass density be anisotropic but diagonal in these coordinates, the time harmonic acoustic equations of state simplify to (the $\exp(+j\omega t)$ convention is used throughout)

$$j\omega\rho_\phi v_\phi = -\frac{1}{r}\frac{\partial p}{\partial \phi}, \tag{3}$$

$$j\omega\rho_r v_r = -\frac{\partial p}{\partial r}, \tag{4}$$

$$j\omega\frac{1}{\lambda}p = -\frac{1}{r}\frac{\partial (r v_r)}{\partial r} - \frac{1}{r}\frac{\partial v_\phi}{\partial \phi}. \tag{5}$$

Now consider the z-invariant 2D Maxwell equations for transverse electric (TE) polarization in cylindrical coordinates with anisotropic but diagonal permittivity and permeability tensors

$$j\omega\mu_r(-H_r) = -\frac{1}{r}\frac{\partial(-E_z)}{\partial \phi}, \tag{6}$$

$$j\omega\mu_\phi H_\phi = -\frac{\partial(-E_z)}{\partial r}, \tag{7}$$

$$j\omega\epsilon_z(-E_z) = -\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r} - \frac{1}{r}\frac{\partial(-H_r)}{\partial\phi}. \tag{8}$$

The above equations have been arranged to highlight the exact duality between these two sets of equations under the variable exchange

$$[p, v_r, v_\phi, \rho_r, \rho_\phi, \lambda^{-1}] \leftrightarrow [-E_z, H_\phi, -H_r, \mu_\phi, \mu_r, \epsilon_z]. \tag{9}$$

The boundary conditions are also preserved under this exchange. At a fluid–fluid interface, p and the normal velocity component are continuous, while at an electromagnetic material interface, E_z and the tangential H component are continuous. Thus at a material discontinuity in radius, p and v_r are continuous, which are equivalent to E_z and H_ϕ under this replacement, which in turn are the electromagnetic quantities that must be continuous. Consequently, a solution to the 2D cylindrical Maxwell equations above is also a solution to the 2D acoustic equations under the above replacement. This further implies that any transformation-based medium [1] for electromagnetics, when restricted to 2D and normal incidence, can also be realized acoustically.

To demonstrate this, we consider the cloaking shell described in [1]. This exact equivalence implies that a cylindrical shell containing specific spatial distributions of mass density and bulk modulus can smoothly bend any incident wavefield around the centre of the shell with essentially zero scattering in any direction, including the forward direction. For a 2D acoustic cloak, the required relative mass density elements and bulk modulus are thus [9]

$$\rho_r = \frac{r}{r-R_1}, \qquad \rho_\phi = \frac{r-R_1}{r}, \qquad \lambda^{-1} = \left(\frac{R_2}{R_2-R_1}\right)^2 \frac{r-R_1}{r}, \tag{10}$$

where R_1 is the radius of the cloaked region and R_2 is the outer edge of the cloaking shell. These values are relative to the bulk modulus and mass density of the fluid outside the shell.

This equivalence implies that the 2D electromagnetic simulations of plane wave interaction with a cloaking shell presented in [9] are also solutions to the analogous acoustic cloaking problems. But, in order to independently and unambiguously demonstrate the validity of this acoustic cloaking approach, a time harmonic 2D cartesian grid finite difference code was written and used to solve the acoustic equations (1) and (2) with an anisotropic mass tensor. Note that the mass density tensor is not diagonal in cartesian coordinates, and the off-diagonal elements are included in the formulation. The computational domain is shown in figure 1, in which a gaussian beam is launched towards an incompressible cylinder 1.5 wavelengths in diameter. All simulation quantities (excitation frequency, bulk modulus and background fluid density) are normalized to unity, and all domain boundaries are perfectly matched layers to prevent reflections.

Figure 2 shows the simulated pressure field for three scenarios. The left panel shows the baseline pressure field with no scatterer. The middle panel shows the pressure field with the uncloaked scatterer, in which strong scattering in all directions is clear, especially the forward direction which produces the shadow region. In contrast, when a 0.75 wavelength thick cloaking shell specified by (10) is placed around the rigid scatterer, the incident wave is smoothly directed around the scatterer with significantly reduced scattering in all directions. 2D acoustic cloaking is thus realizable in an exact analogue to the electromagnetic equivalent.

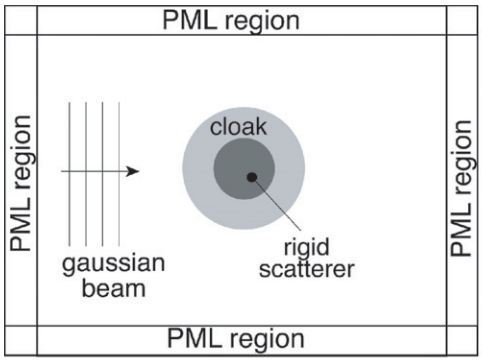

Figure 1. The 2D computational domain, in which an acoustic gaussian beam is incident on an cylindrical object.

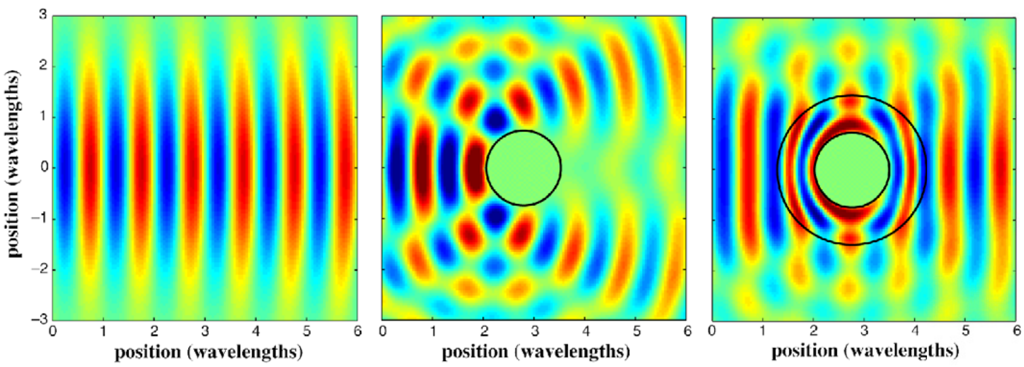

Figure 2. Left panel: The acoustic pressure p in the absence of a scatterer. Middle: The acoustic pressure with an incompressible cylindrical scatterer. Right panel: The acoustic pressure with the same scatterer surrounded by the cloaking shell described in the text.

Some scattering is seen which we attribute to numerical artifacts associated with the cartesian grid approximation of circular surfaces and to the relatively coarse grid used to compute these solutions (20 grid points per wavelength). Although more exact cloaking has been simulated numerically using a conformal triangular mesh [9], clear and effective cloaking performance can also be achieved on a much simpler grid. This confirms the general insensitivity of the ideal cloaking solution to material and geometric imperfections reported in [9], and also suggests that effective cloaks (either electromagnetic or acoustic) could be designed based on a square grid, rather than a conformal one such as that used in [10], which may be simpler to fabricate.

Although anisotropic fluid mass is not a property of ordinary fluids, there is hope for an experimental realization of this idea. Milton *et al* [11] describes conceptually how it can be achieved with spring loaded masses in a manner essentially equivalent to how self resonant electromagnetic metamaterial structures can be used to achieve anisotropic electromagnetic permittivity or permeability of almost any value [14]. Acoustic cloaking realized in this way will consequently be of limited bandwidth.

If imperfect cloaking performance is tolerable, then an additional degree of freedom in the material properties can be obtained in a reduced 2D acoustic cloak that is an exact analogue of the 2D electromagnetic case [9]. In a reduced cloak, the wavenumber is correct but the impedance is not and is thus an ideal cloak only in the short wavelength limit. This extra degree of freedom can be used to eliminate the inhomogeneity of one component of the material, as was done by Schurig *et al* [10] in their experimental demonstration of electromagnetic cloaking. By recasting the acoustic equations (1) and (2), this reduced approach can also shift the needed anisotropy from the mass to the bulk modulus, which may be simpler to realize.

Substantial recent progress has also been made in the experimental realization of acoustic metamaterials. It was described theoretically how a fluid loaded with particles can be used to realize almost any isotropic effective mass and bulk modulus [15], and it has been experimentally demonstrated that 1D acoustic waveguide loaded with resonators can yield an effective bulk modulus of any value, even negative [16]. To be sure, there will be challenges in realizing a functioning approximation of a fluid with continuously variable bulk modulus and anisotropic mass elements, but it appears that this recipe for a 2D acoustic cloaking shell could be physically realizable.

It is interesting to consider why this limited acoustic-electromagnetic equivalence is not evident in the complete elastodynamic analysis of [11]. Here, we derive the equivalence from first principles and show that it is specific to a 2D field in a 3D space. The 2D acoustic fields are described by

$$p = p(x, y), \tag{11}$$

$$\mathbf{v} = v_x(x, y)\hat{\mathbf{x}} + v_y(x, y)\hat{\mathbf{y}}, \tag{12}$$

where p is the pressure and $\mathbf{v}$ is the fluid velocity. We further assume that neither the transformation matrix nor the material property tensors mix x–y coordinates with z so that the transformation matrix has the following form

$$(\Lambda_{i'i}) = \begin{pmatrix} \Lambda_{1'1} & \Lambda_{1'2} & 0 \\ \Lambda_{2'1} & \Lambda_{2'2} & 0 \\ 0 & 0 & \Lambda_{3'3} \end{pmatrix}. \tag{13}$$

In this case, we can show that elastic fluid wave equations (1) and (2), when placed in time harmonic form, are isomorphic with the Maxwell curl equations

$$\nabla \times \mathbf{E} + j\omega\mathbf{B} = \mathbf{0}, \tag{14}$$

$$\nabla \times \mathbf{H} - j\omega\mathbf{D} = \mathbf{0}, \tag{15}$$

and thus form invariant by making the following variable exchange

$$[\mathbf{E}, \mathbf{D}, \mathbf{H}, \mathbf{B}] \leftrightarrow [p\hat{\mathbf{z}}, \lambda^{-1}p\hat{\mathbf{z}}, \mathbf{v}\times\hat{\mathbf{z}}, (\boldsymbol{\rho}\mathbf{v})]. \tag{16}$$

Note that the Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{H} = p\mathbf{v}$ correctly represents power flow in both systems. Making the above substitutions into (14) yields

$$\nabla \times (p\hat{\mathbf{z}}) + j\omega(\boldsymbol{\rho}\mathbf{v})\times\hat{\mathbf{z}} = 0 \tag{17}$$

and applying a vector identity this becomes

$$\nabla p \times \hat{\mathbf{z}} + p\nabla \times \hat{\mathbf{z}} + j\omega\,(\boldsymbol{\rho}\mathbf{v})\times\hat{\mathbf{z}} = 0, \tag{18}$$

which reduces to (1) since $\hat{\mathbf{z}}$ is a constant, and neither ∇p nor $\boldsymbol{\rho}\mathbf{v}$ have z-components. Following the same procedure with (15) yields

$$\mathbf{v}\nabla \times \hat{\mathbf{z}} - \hat{\mathbf{z}}\nabla \times \mathbf{v} + (\hat{\mathbf{z}} \times \nabla)\,\mathbf{v} - (\mathbf{v} \times \nabla)\,\hat{\mathbf{z}} - j\omega\lambda^{-1}p\hat{\mathbf{z}} = 0, \tag{19}$$

which reduces to (2) since $\hat{\mathbf{z}}$ is a constant and $\mathbf{v}$ does not depend on z.

Since we know how μ and ε behave under transformations [1], we can determine how $\boldsymbol{\rho}$ and λ must behave. Applying the substitution (16) to the constitutive relation $\mathbf{B} = \mu\mathbf{H}$ yields

$$(\boldsymbol{\rho}\mathbf{v}) \times \hat{\mathbf{z}} = \mu(\mathbf{v} \times \hat{\mathbf{z}}), \tag{20}$$

and in component form for the primed and unprimed coordinate system

$$\epsilon_{ij3}\rho_{jk}v_k = \mu_{im}\epsilon_{mn3}v_n, \tag{21}$$

$$\epsilon_{i'j'3}\rho_{j'k'}v_{k'} = \mu_{i'm'}\epsilon_{m'n'3}v_{n'}. \tag{22}$$

We substitute the transformation properties of $\mathbf{v}$, (a vector),

$$v_{k'} = \Lambda_{k'i}v_i, \tag{23}$$

and μ, (a tensor density of weight +1),

$$\mu_{i'm'} = \det(\Lambda)^{-1}\Lambda_{i'i}\Lambda_{m'm}\mu_{im}, \tag{24}$$

into (21) and (22) to obtain

$$\epsilon_{i'j'3}\rho_{j'k'}\Lambda_{k'p}v_p = \det(\Lambda)^{-1}\Lambda_{i'i}\Lambda_{m'm}\mu_{im}\epsilon_{m'n'3}\Lambda_{n'q}v_q. \tag{25}$$

Solving (21) and (25) we find the general transformation for the density

$$\rho_{i'j'} = \det(\Lambda)^{-1}(\epsilon_{i'k'3}\Lambda_{k'k}\epsilon_{ki3})(\epsilon_{j'l'3}\Lambda_{l'l}\epsilon_{lj3})\,\rho_{ij}. \tag{26}$$

Similarly, we can apply (16) to the other constitutive relation $\mathbf{D} = \varepsilon\mathbf{E}$ to obtain

$$\lambda^{-1}p\hat{\mathbf{z}} = \varepsilon p\hat{\mathbf{z}}, \tag{27}$$

which reduces to

$$\lambda^{-1} = \varepsilon_{33}.$$

and

$$\varepsilon_{3'3'} = \det(\Lambda)^{-1}\Lambda_{3'i}\Lambda_{3'j}\varepsilon_{ij}, \tag{28}$$

since ε is also a tensor density of weight +1. From the form of the transformation matrix (13) we find that the general transformation for the bulk modulus is

$$\lambda' = \det(\Lambda)\Lambda_{3'3}^{-2}\lambda. \tag{29}$$

Through the above procedure, the 2D acoustic wave has been placed in exact correspondence with one polarization (TE, although it could also have been transverse magnetic) of the 3D electromagnetic equations. A more general equivalence is not possible for acoustics because we need three field variable components for correspondence with each electromagnetic polarization, and a fluid provides only four total with the velocity vector and the scalar pressure. Even if we consider a more general elastic medium in which strain is anisotropic, an exact correspondence to electromagnetics in 3D is not possible as can be seen by examining the energy density, u. For isotropic and anisotropic elastic waves we have respectively

$$u = \frac{1}{2}\lambda\sigma_{ii}\sigma_{kk} + \frac{1}{2}\rho_{mn}v_m v_n, \tag{30}$$

$$u = \frac{1}{2}c_{ijkl}\sigma_{ij}\sigma_{kl} + \frac{1}{2}\rho_{mn}v_m v_n, \tag{31}$$

neither of which is isomorphic with the electromagnetic energy expression

$$u = \frac{1}{2}\varepsilon_{ij}E_i E_j + \frac{1}{2}\mu_{kl}H_k H_l. \tag{32}$$

There may be more special cases of electromagnetic and elastodynamic equivalence, but as shown in [11] and above, a general 3D equivalence is not possible.

In summary, we have shown that the exact equivalence in 2D between fluid acoustics and single polarization electromagnetics holds for anisotropic materials. This anisotropic isomorphism enables all classes of 2D transformation-based solutions of the Maxwell equations to be realized acoustically as well, including the cloaking solution first described in [1]. It was shown numerically that a fluid with an inhomogeneous bulk modulus and with an anisotropic (but diagonal in cylindrical coordinates) and inhomogeneous mass density can smoothly and reflectionlessly direct compressional acoustic waves around a strong scatterer. Engineering a material shell to have the required properties may be a challenge, but it is experimentally feasible in principle.

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