The Nearly Perfectly Matched Layer is a Perfectly Matched Layer

Wenyi Hu and Steven A. Cummer, *Senior Member, IEEE*

***Abstract*—An absorbing boundary condition for computational electromagnetics called the nearly perfectly matched layer (NPML) was recently introduced. This boundary condition, designed to be simple to implement in complex media, was regarded as not quite a perfectly matched layer because its formulation deviates from the standard PML through an inexact variable change. Analytical and numerical results are presented here that show that the NPML is equivalent to the standard perfectly matched layer (PML) in Cartesian coordinates for one-dimensional, two-dimensional, and three-dimensional problems through simple variable changes. Consequently, the NPML performs identically to the PML as an absorbing boundary condition while maintaining its simplicity of implementation.**

***Index Terms*—Complex stretched coordinate, nearly perfectly matched layer (NPML), perfectly matched layer (PML), variable change.**

I. INTRODUCTION

To simulate electromagnetic wave propagation in unbounded domain, absorbing boundary conditions (ABC) have to be applied to eliminate nonphysical reflections from the edges of the computational domain. A variety of effective ABCs are available for such problems [1]–[3]. Bérenger devised the perfectly matched layer (PML) absorbing boundary conditions about a decade ago [2]. Since then, this effective and flexible ABC has been applied extensively to computational electromagnetics. Alternative formulations of the PML were also introduced to simplify the implementation of PML [4], [5]. The original PML was extended to three-dimensional (3-D)[6], [7], dispersive, and anisotropic medium problems [8], [9].

The PML formulation analyzed in this work was initially proposed by Cummer [10] and is referred to as the NPML. The primary advantage of the NPML formulation is its simplicity in implementation because it does not modify the form of the Maxwell equations, even in anisotropic materials. Only several auxiliary variables and the same number of ordinary differential equations must be added to complete the system of equations. The implementation is also straightforward in simple materials. However, the variable change upon which the NPML is based is not exact if the PML conductivity is spatial variant, as it would be in a practical implementation. This did not seem affect the NPML performance in practical applications [10]. In this work, mathematical analysis will prove NPML is fundamentally equivalent to the original PML in Cartesian coordinates. Even for spatially varying PML conductivity profiles, waves reflected from NPML and standard PML layers are identical and, thus, the implementation simplicity of the NPML does not require a trade of exactness or performance.

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The authors are with the Department of Electrical and Computer Engineering, Duke University, Durham, NC 27708 USA (e-mail: wyhu@ee.duke.edu; cummer@ee.duke.edu).

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II. FORMULATIONS AND MATHEMATICAL ANALYSIS

The formulation of the NPML is similar to the original complex stretched coordinate PML formulation [5]. Both are briefly described below.

A. One-Dimensional (1-D) Problem

Here we consider a 1-D problem where fields vary in the x direction in free space. The time harmonic system is governed by the 1-D Maxwell equations

$$\begin{cases} j\omega\varepsilon_0 E_y = -\dfrac{\partial H_z}{\partial x} & (1)\\ j\omega\mu_0 H_z = -\dfrac{\partial E_y}{\partial x}. & (2)\end{cases}$$

By using complex stretched coordinate approach, the standard PML can be written as

$$\begin{cases} j\omega\varepsilon_0 E_y = -\dfrac{1}{1+\frac{\sigma_x}{j\omega}}\dfrac{\partial H_z}{\partial x} & (3)\\ j\omega\mu_0 H_z = -\dfrac{1}{1+\frac{\sigma_x}{j\omega}}\dfrac{\partial E_y}{\partial x} & (4)\end{cases}$$

where σ_x is the PML conductivity.

The governing equations for the NPML are [10]

$$\begin{cases} j\omega\varepsilon_0 E_y = -\dfrac{\partial}{\partial x}\left(\dfrac{1}{1+\frac{\sigma_x}{j\omega}}H_z\right) & (5)\\ j\omega\mu_0 H_z = -\dfrac{\partial}{\partial x}\left(\dfrac{1}{1+\frac{\sigma_x}{j\omega}}E_y\right). & (6)\end{cases}$$

The two systems above are identical if σ_x is spatial-invariant. However, in practical applications, the PML conductivity varies with position to minimize numerical reflections from PML layers. Cummer [10] concluded (5) and (6) are, thus, not an exact PML, although their performance in an absorbing boundary condition is as good as the standard PML.

These two systems are in fact equivalent even if the PML conductivity is spatially varying. First, we rearrange system (5) and (6) to

$$\begin{cases} j\omega\varepsilon_0 E_y = -\dfrac{\partial}{\partial x}\overline{H}_z & (7)\\ j\omega\mu_0 H_z = -\dfrac{\partial}{\partial x}\overline{E}_y & (8)\end{cases}$$

where

$$\begin{cases} \overline{H}_z = \dfrac{1}{1+\frac{\sigma_x}{j\omega}} H_z & (9) \\ \overline{E}_y = \dfrac{1}{1+\frac{\sigma_x}{j\omega}} E_y. & (10) \end{cases}$$

Note that [10] used the notation $\widetilde{\bullet}$ to describe the same variable change. We use $\overline{\bullet}$ here to distinguish it from a different variable change (which is often denoted by $\widetilde{\bullet}$) in the stretched field standard PML. Substitute (9) and (10) into (7) and (8), then we obtain

$$\begin{cases} j\omega\varepsilon_0 \left(1+\dfrac{\sigma_x}{j\omega}\right) \overline{E}_y = -\dfrac{\partial}{\partial x}\overline{H}_z & (11) \\ j\omega\mu_0 \left(1+\dfrac{\sigma_x}{j\omega}\right) \overline{H}_z = -\dfrac{\partial}{\partial x}\overline{E}_y & (12) \end{cases}$$

i.e.,

$$\begin{cases} j\omega\varepsilon_0 \overline{E}_y = -\dfrac{1}{1+\frac{\sigma_x}{j\omega}}\dfrac{\partial}{\partial x}\overline{H}_z & (13) \\ j\omega\mu_0 \overline{H}_z = -\dfrac{1}{1+\frac{\sigma_x}{j\omega}}\dfrac{\partial}{\partial x}\overline{E}_y. & (14) \end{cases}$$

The NPML (13) and (14) and the standard PML (3) and (4) are in exactly the same form if we replace $\overline{E}_y$, $\overline{H}_z$ in (13) and (14) with E_y, H_z. Thus, $\overline{E}_y$ and $\overline{H}_z$ in the NPML are equivalent to E_y and H_z in standard PML, and E_y and H_z in NPML are equivalent to $\widetilde{E}_y$ and $\widetilde{H}_z$ in standard PML, where $(1+\sigma_x/j\omega)E_y = \widetilde{E}_y$ and $(1+\sigma_x/j\omega)H_z = \widetilde{H}_z$. Since E_y and H_z are continuous across a standard PML interface, $\overline{E}_y$ and $\overline{H}_z$ must be continuous across an NPML interface (and must be coded this way). Therefore, auxiliary fields in NPML are not reflected because they are governed by same (13)–(14) as normal fields in a standard PML. Furthermore, auxiliary fields and normal fields in NPML are related through (9)–(10). Consequently, normal fields are also not reflected from the NPML interface. Aside from these variable shifts, the standard PML and the NPML are exactly the same, and thus the reflections from a standard PML and an NPML layer are identical, even if σ_x is spatially varying. And despite the apparently different variable changes in ((3)–(4)) and ((5)–(6)), the NPML is theoretically a PML. This point will be illuminated by the numerical simulation which follows.

B. Two-Dimensional (2-D) Problem

A similar analysis shows that the NPML is also equivalent to the standard PML in 2-D Cartesian coordinates. For simplicity without loosing generality, we consider 2-D TE fields in free space. The fields are invariant in the z direction. Thus, the Maxwell equations are reduced to three partial differential equations only involving three field components E_x, E_y, and H_z. The NPML that absorbs in x and y is expressed as

$$\begin{cases} j\omega\mu_0 H_z = \dfrac{\partial}{\partial y}\overline{E}_x^y - \dfrac{\partial}{\partial x}\overline{E}_y^x & (15) \\ j\omega\varepsilon_0 E_y = -\dfrac{\partial}{\partial x}\overline{H}_z^x & (16) \\ j\omega\varepsilon_0 E_x = \dfrac{\partial}{\partial y}\overline{H}_z^y & (17) \end{cases}$$

$$\begin{cases} \overline{E}_x^y = \dfrac{1}{1+\frac{\sigma_y}{j\omega}} E_x & (18) \\ \overline{E}_y^x = \dfrac{1}{1+\frac{\sigma_x}{j\omega}} E_y & (19) \\ \overline{H}_z^x = \dfrac{1}{1+\frac{\sigma_x}{j\omega}} H_z & (20) \\ \overline{H}_z^y = \dfrac{1}{1+\frac{\sigma_y}{j\omega}} H_z. & (21) \end{cases}$$

Substituting (18)–(21) into (15)–(17), then we can reconstruct the NPML system as

$$\begin{cases} j\omega\mu_0 \overline{\overline{H}}_z = \dfrac{1}{1+\frac{\sigma_y}{j\omega}}\dfrac{\partial}{\partial y}\overline{\overline{E}}_x - \dfrac{1}{1+\frac{\sigma_x}{j\omega}}\dfrac{\partial}{\partial x}\overline{\overline{E}}_y & (22) \\ j\omega\varepsilon_0 \overline{\overline{E}}_y = -\dfrac{1}{1+\frac{\sigma_x}{j\omega}}\dfrac{\partial}{\partial x}\overline{\overline{H}}_z & (23) \\ j\omega\varepsilon_0 \overline{\overline{E}}_x = \dfrac{1}{1+\frac{\sigma_y}{j\omega}}\dfrac{\partial}{\partial y}\overline{\overline{H}}_z & (24) \end{cases}$$

where $(\overline{\overline{\bullet}})$ denotes the corresponding field value multiplied by the stretched coordinate factor in both directions, i.e., $1/(1+\sigma_x/j\omega)\cdot 1/(1+\sigma_y/j\omega)$.

Comparing the system (22)–(24) with the system governing the standard PML

$$\begin{cases} j\omega\mu_0 H_z = \dfrac{1}{1+\frac{\sigma_y}{j\omega}}\dfrac{\partial}{\partial y}E_x - \dfrac{1}{1+\frac{\sigma_x}{j\omega}}\dfrac{\partial}{\partial x}E_y & (25) \\ j\omega\varepsilon_0 E_y = -\dfrac{1}{1+\frac{\sigma_x}{j\omega}}\dfrac{\partial}{\partial x}H_z & (26) \\ j\omega\varepsilon_0 \dot{E}_x = \dfrac{1}{1+\frac{\sigma_y}{j\omega}}\dfrac{\partial}{\partial y}H_z & (27) \end{cases}$$

we find they are in same form except that all the fields in the NPML system are multiplied by the stretched coordinate factors in x and y direction. The NPML and standard PML systems are thus identical under this variable change. Similar to the 1-D case, $\overline{\overline{H}}_z$ is continuous across NPML boundaries while, in the standard PML, H_z is continuous across PML boundaries. The fields reflected from identically inhomogeneous NPML and standard PML domains must be identical in 2-D as well. It can be shown that this conclusion remains true even in 3-D problems through a variable change involving coordinate stretch factors in all three dimensions. The NPML can also be extended to cylindrical and spherical coordinates straightforwardly and we are currently investigating its equivalence to the standard PML in these coordinate systems.

III. Numerical Experiments

A. Oblique Incident Plane Wave

To validate the above analysis, we compare the performance of the NPML and the standard PML for a 2-D obliquely incident plane wave (fields invariant in z). We also will confirm the variable switching in NPML layers. We consider a standard PML and an NPML that absorb in the $+x$ direction, and in y we use periodic boundary conditions. A sinusoidal source H_{zs} with frequency $\omega = 2\pi \times 300$ MHz is applied on the $x = 0$ plane. The source phase varies in y to excite an obliquely incident plane

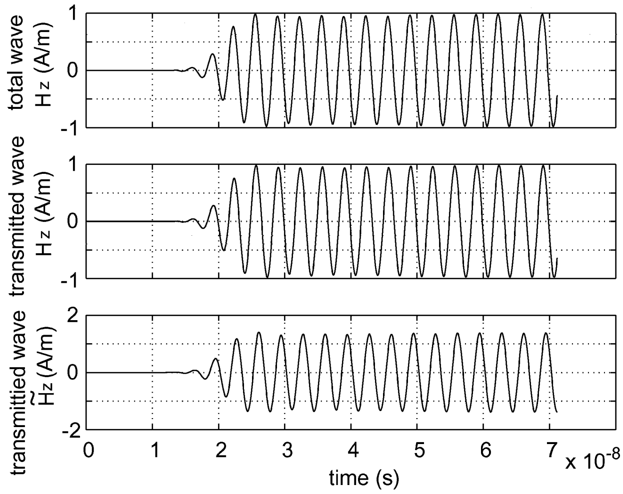

Fig. 1. Field values at the observations location in obliquely incident plane wave case with infinite standard PML and constant conductivity. Top panel: total field on free-space side (half-cell away from the interface). Middle panel: normal transmitted field on the PML side (half-cell away from the interface). Bottom panel: auxiliary transmitted field on the PML side (half-cell away from the interface).

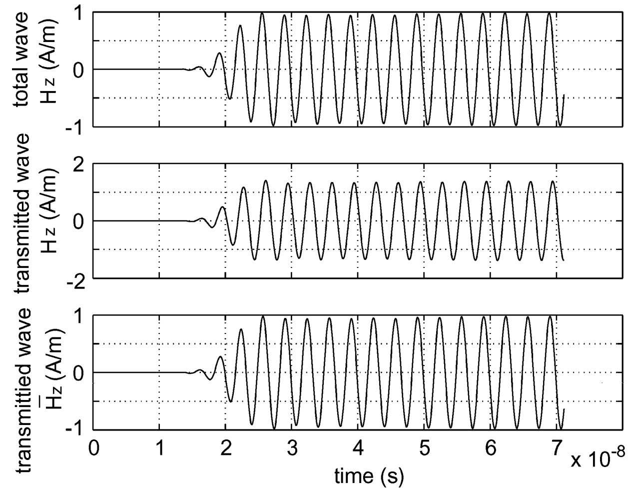

Fig. 2. Field values at the observation locations in obliquely incident plane wave case with infinite NPML and constant conductivity. Top panel: total field on free-space side (half-cell away from the interface). Middle panel: normal transmitted field on the NPML side (half-cell away from the interface). Bottom panel: auxiliary transmitted field on the NPML side (half-cell away from the interface).

wave. The NPML layer has constant conductivity and is thick enough to eliminate reflection at the right PEC boundary of the computation domain. The angle of incidence on the PML (θ) is 30°, namely

$$H_{zs}(y,t) = \begin{cases} \sin\left(\omega t + \frac{y\omega \sin\theta}{c}\right) \exp\left[-\left(\frac{t-t_0}{\delta}\right)^2\right] & t \le t_0 \\ \sin\left(\omega t + \frac{y\omega\sin\theta}{c} y\right) & t > t_0 \end{cases}. \tag{28}$$

The slow Gaussian turn-on in (28) is used to minimize the transient high-frequency components.

Because we are trying to inspect how the variable switching scheme mentioned in previous section behaves in NPML in numerical simulations, we locate one sample point recording the total fields on the free-space side and the other sample point recording the transmitted fields on the PML side (both are half-cell away from the interface). The simulation results are shown in Fig. 1 (standard PML) and Fig. 2 (NPML).

We find the field values on the free-space side for standard PML case and NPML cases are identical. However, the transmitted wave H_z in standard PML is identical to $\overline{H}_z$ in NPML, while the transmitted wave H_z in NPML is identical to $\widetilde{H}_z$ ($\widetilde{H}_z$ is the auxiliary variable in the implementation of standard PML). This is not surprising because $(1+\sigma_x/j\omega)\overline{H}_z = H_z$ in NPML and $(1+\sigma_x/j\omega)H_z = \widetilde{H}_z$ in standard PML. Obviously, the variable switching does not affect the wave propagation behavior in the non-PML region, but it does modify the fields in the PML layers. The numerical results are fully consistent with the mathematical analysis in the second section.

B. 2-D Problem With Line Source

We now confirm the equivalence of the NPML and standard PML in a more complicated scenario with a pulsed source and inhomogeneous PML conductivity. The computation domain is a 0.6 m × 0.6 m square. The cell size is 4 mm × 4 mm. A hard magnetic field infinite line source is located at the center of the computation domain. This line source radiates a broadband Gaussian pulse. We record the fields near the interface between the free space and the NPML layer. The conductivity profiles for NPML and standard PML are same spatially varying function

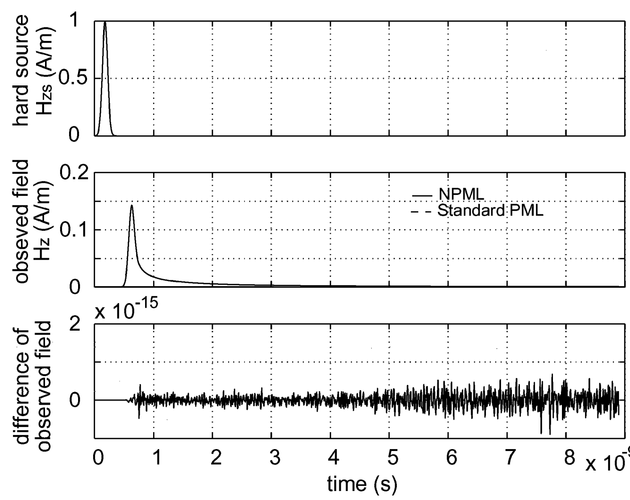

Fig. 3. 2-D Cartesian coordinate simulations. Top panel: hard source located at the center of the computation domain. Middle panel: observed field values at the sample point (in free-space region and half-cell away from the interface) for NPML and standard PML simulations. Bottom panel: the field value difference between NPML and standard PML simulations shown in middle panel.

$$\sigma(\xi) = \begin{cases} \sigma_m \left[\frac{(L_{pml}-\xi)}{L_{pml}}\right]^2 & \xi \le L_{pml} \\ 0 & L_{pml} < \xi \le L_{pml}+\xi_m \\ \sigma_m \left[\frac{(\xi - L_{pml}-\xi_m)}{L_{pml}}\right]^2 & \xi \le L_{pml} \end{cases} \tag{29}$$

where $\xi \in \{x, y\}$, L_{pml} is the thickness of the PML layer, σ_m is the maximum conductivity and $\xi_m = 0.6$ m is the regular medium size.

The fields reflected by each PML are shown in Fig. 3. The difference between the NPML and the standard PML is simply accumulated double precision truncation error. As shown analytically, we therefore conclude that the NPML and standard PML are identical even with spatially varying PML conductivity and in fully 2-D corner regions.

IV. Conclusion

In this work, the NPML has been shown analytically to be exactly equivalent to the standard PML in Cartesian coordinates for 1-D, 2-D, and 3-D problems. This is somewhat surprising in light of the similar but not identical variable changes involved in the derivation of each. Numerical results confirm that the fields reflected from identically inhomogeneous NPML and standard PML layers are identical. The difference between these two PML forms is only in implementation. As argued in [10], the NPML is particularly simple to implement because it retains the form of the regular medium partial difference equations even in lossy and anisotropic materials and does so with a minimum of auxiliary variables and additional computation. These properties make the NPML an easy to implement and flexible absorbing boundary condition that can be applied to a wide variety of wave propagation and scattering problems.

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