A Simple, Nearly Perfectly Matched Layer for General Electromagnetic Media

Steven A. Cummer, *Senior Member, IEEE*

***Abstract*—A new implementation of the perfectly matched layer (PML) absorbing boundary condition is presented. This formulation is designed such that the partial differential equations in the PML are identical to those in the regular medium for any linear electromagnetic material. This makes this method particularly simple to implement, especially in dispersive and anisotropic materials. We call this method the nearly perfectly matched layer (NPML) because it employs variable changes that are not strictly exact when the PML conductivity is spatially variant. Comparisons with the convolutional PML in a Lorentz dielectric show that the NPML is as effective an absorber as exact PML formulations.**

***Index Terms*—Absorbing boundary condition, perfectly matched layer (PML).**

I. Introduction

SINCE the publication of Berenger's original split-field perfectly matched layer (PML) [1], this highly effective absorbing boundary condition has been adapted in a variety of ways. Alternate PML formulations have been reported [2], [3], [4] that all have similar broadband absorbing performance and efficiency as the original PML. Formulations have also been derived that are effective broadband absorbers for more than just the simple dielectrics of the original PML [5], [6], [7].

We report here a PML formulation that, like others [6], [7], is sufficiently general to be applied to essentially any linear medium, either conducting, dispersive, anisotropic, or with a combination these properties. The formulation presented here has some advantages over previous formulations, particularly in ease of implementation. It also has some disadvantages that do not appear to affect its performance in practice. We call it the nearly perfectly matched layer (NPML) for reasons described below.

The fundamental NPML partial differential equations are *exactly* the same as those in the simulation space for any linear material. For simple dielectrics, this is not an important advantage. But for a complicated material, particularly an anisotropic one, explicit time-centered finite difference equations can be tedious to derive because all of the equations must be solved simultaneously. Any additional variables that appear in the PML partial differential equations require a rederivation of the full difference system. With the NPML, the difference equations for the PML region can be copied exactly from the regular medium, independent of the differencing scheme used. Only simple ordinary differential equations need to be added to complete the NPML field equations.

The NPML is also based purely on differential equations (as opposed to convolutional approaches). It can, thus, be applied directly and simply to any class and order of finite difference approximation. Applying the NPML to cylindrical and spherical coordinate systems is also straightforward [8]. The computational efficiency of the NPML is the same as the convolutional PML (CPML) [7] and essentially the same as other PML formulations. Modified PML formulations like the complex frequency shifted (CFS) PML [9] can be also implemented within the NPML framework.

This method has some differences from other PML formulations that do not appear to be of practical importance. The NPML, like the original split-field and many other formulations, is only weakly well-posed [10]. Although weak well-posedness can, in theory, lead to instability, this does not occur in practice under ordinary circumstances.

Theoretically, NPML is perfectly matched for all incident angles only when the PML conductivity is spatially invariant. For practical reasons, PML conductivity is always implemented inhomogeneously and the NPML is not strictly exact. For this reason, we call this method the nearly perfectly matched layer. We note that other PML formulations have been reported and found effective that are similarly inexact for spatially varying PML conductivity [11]. We show in simulations below that this imperfection is not significant and that it performs as an absorbing boundary condition (ABC), as well as a strictly perfectly matched layer. This is not surprising since numerical discretization, not theoretical exactness, keeps the PML from being a perfect ABC. We surmise that NPML performance is limited by the same numerical issues, not the slightly inexact formulation.

II. Derivation

The NPML is based on the standard stretched coordinate PML formulation [12] but deviates with the goal of preserving the same fundamental form for the PML partial differential equations. As an example, consider the two-dimensional (2-D) TM Maxwell's equations, assuming $\partial/\partial y = 0$ and an $\exp(j\omega t)$ time variation, which include material responses through current density J and magnetization M

$$\frac{\partial E_y}{\partial z} = j\omega\mu_0 H_x + j\omega\mu_0 M_x \quad (1)$$

$$\frac{\partial E_y}{\partial x} = -j\omega\mu_0 H_z - j\omega\mu_0 M_z \quad (2)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon_0 E_y + J_y. \quad (3)$$

Manuscript received July 3, 2002; revised September 26, 2002. This work was supported by NASA Grant NAG5-10270 and U.S. Army Research Office gGrant DAAD19-01-1-0565. The review of this letter was arranged by Associate Editor Dr. Shigeo Kawasaki.

The author is with the Electrical and Computer Engineering Department, Duke University, Durham, NC 27708 USA (e-mail: cummer@ee.duke.edu).

Digital Object Identifier 10.1109/LMWC.2003.810124

Essentially, all complicated materials (such as simple conductors, plasmas, and dispersive electric or magnetic materials) fit this model where ordinary differential equations relate magnetization $\mathbf{M}$ with magnetic field $\mathbf{H}$, and electric current density $\mathbf{J}$ with electric field $\mathbf{E}$ [13] (recall that electric current density $\mathbf{J}$ and electric polarization $\mathbf{P}$ are equivalent through $\mathbf{J} = \partial P/\partial t$). Although the electromagnetic properties of most materials are commonly expressed in the frequency domain, this frequency domain relationship is originally derived from a time domain model. For anisotropic materials, M and J (or P) may be functions of non-TE fields, and handling them with this method would simply require including the equations for all field components.

To derive a PML that absorbs waves propagating in the $+z$ direction, we apply complex coordinate stretching [12] in which $\partial z \Rightarrow \partial \tilde{z} = (1 + \sigma_z(z)/j\omega)\,\partial z$. Applying this coordinate change to (1)–(3) and redefining some variables gives the system of equations in real coordinates

$$\frac{\partial \tilde{E}_y}{\partial z} = j\omega\mu_0 H_x + j\omega\mu_0 M_x \tag{4}$$

$$\frac{\partial E_y}{\partial x} = -j\omega\mu_0 H_z - j\omega\mu_0 M_z \tag{5}$$

$$\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\epsilon_0 E_y + J_y \tag{6}$$

and

$$\tilde{F} = \left(1 + \frac{\sigma_z}{j\omega}\right)^{-1} F, \qquad \text{where } F \in \{E_y,\, H_x\}. \tag{7}$$

In other words, $(\tilde{\cdot})$ denotes multiplication of the physical fields by the stretched coordinate factor $(1 + \sigma_z/j\omega)^{-1}$. This variable change is not strictly correct if σ_z is z-dependent. However, we show with simulations below that this approximation does not substantially affect the NPML performance, and it can be shown that for normal PML parameters, the layer is very close to perfectly matched. This will be analyzed in detail in future work.

Note that with the redefinition of the stretched $(\tilde{\cdot})$ variables, the PML partial differential equations (4)–(6) are in exactly the same form as the medium equations (1)–(3).

It is simple to transform this system to the time domain, where they must be coupled to the ordinary differential equations (ODEs)

$$\frac{\partial \tilde{E}_y}{\partial t} + \sigma_z \tilde{E}_y = \frac{\partial E_y}{\partial t} \tag{8}$$

$$\frac{\partial \tilde{H}_x}{\partial t} + \sigma_z \tilde{H}_x = \frac{\partial H_x}{\partial t}. \tag{9}$$

to connect the stretched and unstretched fields. The system is completed by additional ODEs relating M and H and/or J and E that model the specific material response (as shown by the specific example below).

III. Properties of the NPML

Additional variables have been added to the higher order terms in the partial differential equation (PDE) system. This implies weak well-posedness of the system [10], a property that many PML formulations have. Although this can cause instability in theory, it does not appear to be an issue of practical importance in ordinary problems. The NPML equations are PDEs and ODEs, and, thus, can be approximated by any class (second order, higher order, explicit, implicit) of finite difference technique.

The PDEs in the NPML system will always be in exactly the same form as Maxwell's equations in the physical medium, with no additional terms. This means that, independent of differencing scheme, the partial difference equations and coefficients for the NPML can be simply copied from those in the non-PML region. Only a few simple ODEs of a single form [i.e., (8) and (9)] must be added to complete the NPML. This makes the NPML simple to implement for any material. NPML formulations of PML variants, like the complex frequency shifted PML [9], are also easy to derive. Only the NPML ODEs change form.

The derivation above shows explicitly how to generate the NPML method for single direction attenuation in Cartesian coordinates. For corner regions, multiple coordinate directions must be stretched simultaneously, but this is straightforward. For cylindrical or spherical coordinates, coordinate stretching as described in [8] can easily be applied. These more complicated cases will result in more auxiliary variables and ODEs but are fundamentally similar to the derivation above.

IV. Example and Demonstration

To show the effectiveness of the NPML, we model 2D Cartesian ($\partial/\partial y = 0$), linearly polarized TM wave propagation in a Lorentz (resonant) dispersive dielectric. The field equations in such a material are

$$\epsilon_0 \frac{\partial E_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \frac{\partial P_y}{\partial t} \tag{10}$$

$$\mu_0 \frac{\partial H_x}{\partial t} = \frac{\partial E_y}{\partial z} \tag{11}$$

$$\mu_0 \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x} \tag{12}$$

$$\frac{\partial^2 P_y}{\partial t^2} + \nu \frac{\partial P_y}{\partial t} + \omega_0^2 P_y = F\omega_0^2 E_y \tag{13}$$

where ν, ω_0, and F are parameters of the dispersive dielectric. Following the NPML derivation procedure, the z-absorbing NPML equations are

$$\epsilon_0 \frac{\partial E_y}{\partial t} = \frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial H_z}{\partial x} - \frac{\partial P_y}{\partial t} \tag{14}$$

$$\frac{\partial \tilde{E}_y}{\partial t} + \sigma_z \tilde{E}_y = \frac{\partial E_y}{\partial t} \tag{15}$$

$$\frac{\partial^2 P_y}{\partial t^2} + \nu \frac{\partial P_y}{\partial t} + \omega_0^2 P_y = F\omega_0^2 E_y \tag{16}$$

$$\mu_0 \frac{\partial H_x}{\partial t} = \frac{\partial \tilde{E}_y}{\partial z} \tag{17}$$

$$\frac{\partial \tilde{H}_x}{\partial t} + \sigma_z \tilde{H}_x = \frac{\partial H_x}{\partial t} \tag{18}$$

$$\mu_0 \frac{\partial H_z}{\partial t} = -\frac{\partial E_y}{\partial x}. \tag{19}$$

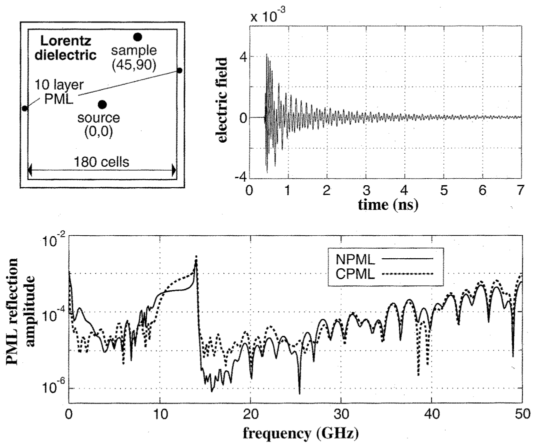

Fig. 1. Top left panel: simulation geometry. Top right panel: electric field waveform after propagation through the Lorentz dielectric described in the text. Bottom panel: overall reflection coefficient magnitude in the Lorentz dielectric of the NPML and the CPML with parameters described in the text. ABC performance is similar.

The resulting system contains the medium PDE/ODE equations (10)–(13), two extra variables, and two extra ODEs. These equations are listed in the order needed to implement an explicit finite difference approximation. At this point, any finite difference scheme (standard second order leapfrog or higher order) could be used to approximate these equations. The x-absorbing and corner NPML equations are similar, with different or additional extra variables but always the same PDE system.

To validate the NPML performance as an absorbing boundary condition, these 2-D equations were discretized using the standard second order leapfrog scheme, placing E and P at integer time steps, and H and $J = \partial P/\partial t$ at half time steps. A 28-ps (e^{-1} full width) Gaussian electric-field pulse is excited at the center of a domain composed entirely of Lorentz dielectric with parameters $F = 2$, $\omega_0 = 2\pi \times 10^{10}$ s^{-1}, and $\nu = 3 \times 10^8$ s^{-1}. The discretization parameters are $\Delta t = 2.33$ ps and $\Delta z = 1.0$ mm.

The simulation geometry is shown in the upper left panel of Fig. 1. The electric field waveform at the simulation edge was recorded in two simulations with ten cell NPML and CPML layers just beyond the sampled point. A much larger reference simulation produced the no-reflections signal to which these are compared. The top right panel of Fig. 1 shows the reference field waveform for perspective. The NPML parameters were $\sigma_{\max} = 3.5 \times 10^{12}$ and $\sigma(z) \propto z^{4.3}$ and the CPML parameters were $\sigma_{\max} = 1.0 \times 10^{12}$ and $\sigma(z) \propto z^{3.2}$. The different PML parameters were necessary to make the PMLs close to optimal and perform similarly.

The bottom panel shows the magnitude of the PML reflection coefficient as a function of frequency, defined as $FFT\,(E_{\rm pml} - E_{\rm ref})/FFT(E_{\rm pml})$. Note that this coefficient includes reflections from all of the boundaries, not just the one nearest to the sample point. The NPML and CPML are comparably effective for the frequency-dependent propagation in a resonant dielectric, with a reflection coefficient of -80 to -90 dB over the most of the incident pulse bandwidth in a ten-layer PML. Reflections are somewhat higher near the dielectric resonance for both methods, but, practically, this is not especially important because this frequency range is strongly attenuated for the chosen dielectric parameters.

Most importantly, this simulation shows that the NPML, which is particularly easy to implement but not quite exact, performs as well as the exact CPML. This indicates that the details of the numerical implementation, rather than the exactness of the analytical formulation, dominate the performance when the PML layers are nearly optimal. We conclude that the NPML is effective, versatile, and simple to apply to any linear electromagnetic material. Further analysis is needed to examine in detail its performance under a wider variety of conditions and materials.

ACKNOWLEDGMENT

The author would like to thank Prof. Q. Liu and Mr. W. Hu of the Duke University Electrical and Computer Engineering Department for valuable discussions.

REFERENCES

[1] J.-P. Berenger, "Numerical reflection from FDTD-PMLs: A comparison of the split PML with the unsplit and CSF PMLs," *IEEE Trans. Antennas Propagat.*, vol. 50, pp. 258–265, Mar. 2002.
[2] Z. S. Sacks, D. M. Kingsland, R. Lee, and J. F. Lee, "A perfectly matched anisotropic absorber for use as an absorbing boundary condition," *IEEE Trans. Antennas Propagat.*, vol. 43, pp. 1460–1463, Dec. 1995.
[3] L. Zhao and A. C. Cangellaris, "GT-PML: Generalized theory of perfectly matched layers and its application to the reflectionless truncation of finite-difference time-domain grids," *IEEE Trans. Microwave Theory*, vol. 44, pp. 2555–2563, Dec. 1996.
[4] R. W. Ziolkowski, "Time-derivative Lorentz material model-based absorbing boundary condition," *IEEE Trans. Antennas Propagat.*, vol. 45, pp. 1530–1535, Oct. 1997.
[5] S. D. Gedney, "An anisotropic PML absorbing media for the FDTD simulation of fields in lossy and dispersive media," *Electromagnetics*, vol. 16, pp. 399–415, 1996.
[6] G. X. Fan and Q. H. Liu, "An FDTD algorithm with perfectly matched layers for general dispersive media," *IEEE Trans. Antennas Propagat.*, vol. 48, pp. 637–646, May 2000.
[7] J. A. Roden and S. D. Gedney, "Convolution PML (CPML): An efficient FDTD implementation of the CFS-PML for arbitrary media," *Microwave Opt. Technol. Lett.*, vol. 27, no. 5, pp. 334–339, 2000.
[8] F. L. Teixeira and W. C. Chew, "PML-FDTD in cylindrical and spherical grids," *IEEE Microwave Guided Wave Lett.*, vol. 7, pp. 285–287, Sept. 1997.
[9] M. Kuzuoglu and R. Mittra, "Frequency dependence of the constitutive parameters of causal perfectly matched anisotropic absorbers," *IEEE Microwave Guided Wave Lett.*, vol. 6, pp. 447–449, Dec. 1996.
[10] S. Abarbanel and D. Gottlieb, "A mathematical analysis of the PML method," *J. Comput. Phys.*, vol. 134, pp. 357–363, 1997.
[11] E. M. Tentzeris, R. L. Robertson, J. F. Harvey, and L. P. B. Katehi, "PML absorbing boundary conditions for the characterization of open microwave circuit components using multiresolution time-domain techniques (MRTD)," *IEEE Trans. Antennas Propagat.*, vol. 47, pp. 1709–1715, Nov. 1999.
[12] W. C. Chew and W. H. Weedon, "A 3D perfectly matched medium from modified Maxwell's equations with stretched coordinates," *Microwave Opt. Technol. Lett.*, vol. 7, no. 13, pp. 599–604, Sept. 1994.
[13] J. L. Young and R. O. Nelson, "A summary and systematic analysis of FDTD algorithms for linearly dispersive media," *IEEE Antennas Propagat. Mag.*, vol. 43, pp. 61–77, Feb. 2001.