Three-level Systems and Two-photon Transitions

ECE 590.01

Quantum Engineering with Atoms

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General Rotating Frame

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum_i \left[\tilde{L}_i \rho \tilde{L}_i^\dagger - \frac{1}{2}\left(\tilde{L}_i^\dagger \tilde{L}_i \rho + \rho \tilde{L}_i^\dagger \tilde{L}_i\right)\right]$$

$$= -\frac{i}{\hbar}[H,\rho] + \sum_i \mathcal{D}[\tilde{L}_i](\rho) \qquad H = H_0 + H_I$$

$$\tilde{\rho} = e^{iH_D t/\hbar} \rho e^{-iH_D t/\hbar} \qquad [H_0, H_D] = 0$$

$$\dot{\tilde{\rho}} = e^{iH_D t/\hbar}\left(i\frac{H_D}{\hbar}\rho + \dot{\rho} - i\rho\frac{H_D}{\hbar}\right)e^{-iH_D t/\hbar}$$

$$= -\frac{i}{\hbar}[H_0 - H_D, \tilde{\rho}] - \frac{i}{\hbar}e^{iH_D t/\hbar}\left(H_I\rho - \rho H_I + \sum_i \mathcal{D}[L_i](\rho)\right)e^{-iH_D t/\hbar}$$

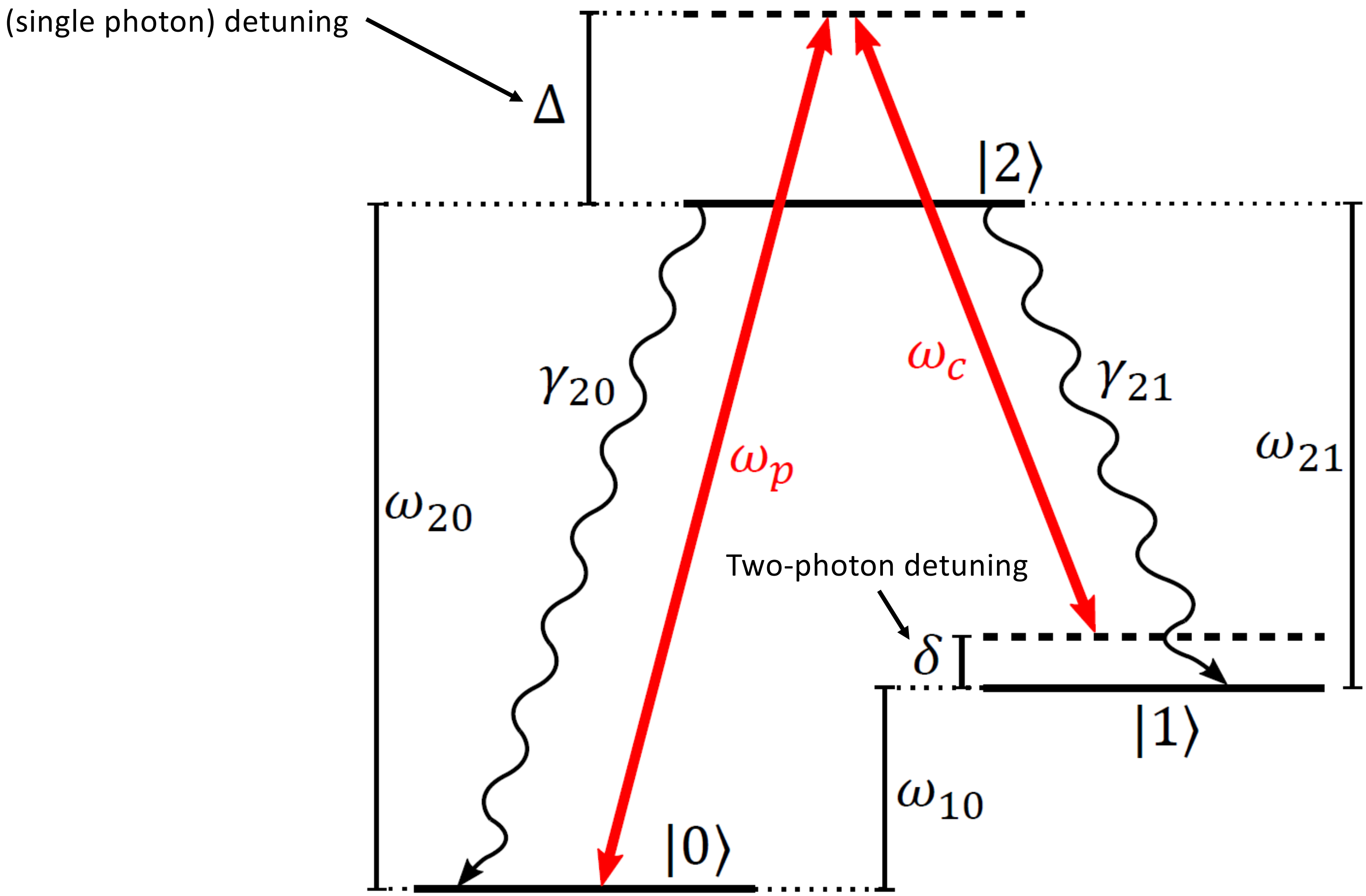
$$= -\frac{i}{\hbar}\left[H_0 - H_D + \tilde{H}_I, \tilde{\rho}\right] + \sum_i \mathcal{D}[\tilde{L}](\tilde{\rho}) \equiv -\frac{i}{\hbar}\left[\tilde{H}, \tilde{\rho}\right] + \sum_i \mathcal{D}[\tilde{L}](\tilde{\rho})$$

Expectation Values in Rotating Frame

$$\begin{aligned}
\langle A \rangle &= \mathrm{Tr}(A\rho) \\
&= \sum_i \sum_j p_j \langle \psi_i | A | \psi_j \rangle \langle \psi_j | \psi_i \rangle \\
&= \sum_i \sum_j p_j \langle \psi_i | e^{-iH_D t/\hbar} e^{iH_D t/\hbar} A e^{-iH_D t/\hbar} e^{iH_D t/\hbar} | \psi_j \rangle \langle \psi_j | e^{-iH_D t/\hbar} e^{iH_D t/\hbar} | \psi_i \rangle \\
&= \sum_i \sum_j p_j \langle \tilde{\psi}_i | \tilde{A} | \tilde{\psi}_j \rangle \langle \tilde{\psi}_j | \tilde{\psi}_i \rangle = \mathrm{Tr}(\tilde{A}\tilde{\rho})
\end{aligned}$$

$$\tilde{A} = \exp(iH_D t/\hbar) A \exp(-iH_D t/\hbar)$$

Three-level System in Harmonic Trap

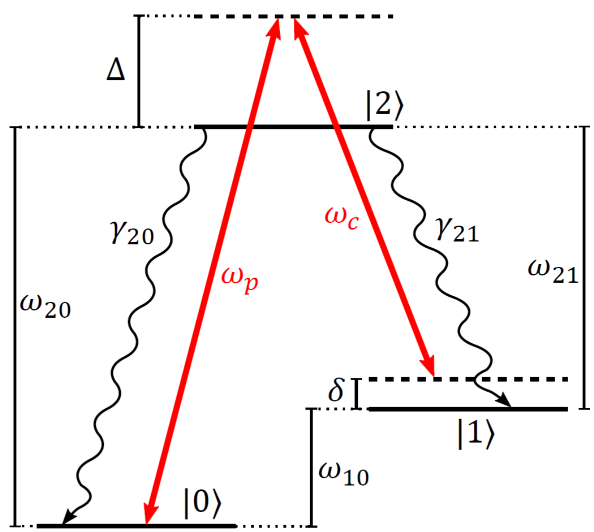


Hamiltonians

$$H_0^{(A)} = \hbar\omega_{10} |1\rangle \langle 1| + \hbar\omega_{20} |2\rangle \langle 2|$$ Atom energy levels

$$H_0^{(M)} = \hbar\nu \left(a^\dagger a + \frac{1}{2} \right)$$ Atom motion in harmonic potential

$$H_I = -q\vec{E} \cdot \hat{\boldsymbol{r}}$$ Atom-light interaction



$$H_I = H_I^{(p)} + H_I^{(c)}$$

Interaction Hamiltonian – Schrodinger Picture

$$H_I^{(p)} = -qE_0^{(p)}\hat{\epsilon}_p \cos(\omega_p t - k_p x + \phi_p)\hat{r}\left(d_{02}\left|0\right\rangle\left\langle 2\right| + d_{02}^*\left|2\right\rangle\left\langle 0\right|\right)$$

$$= \left(\hbar\Omega_p\left|0\right\rangle\left\langle 2\right| + \hbar\Omega_p^*\left|2\right\rangle\left\langle 0\right|\right)\cos(\omega_p t - \vec{k}_p\cdot\vec{x} + \phi_p)$$

$$\hbar\Omega_p = -qE_0^{(p)}d_{02}(\hat{\epsilon}_p\cdot\hat{r})$$

Amplitude Polarization

$$H_I^{(c)} = \left(\hbar\Omega_c\left|1\right\rangle\left\langle 2\right| + \hbar\Omega_c^*\left|2\right\rangle\left\langle 1\right|\right)\cos(\omega_c t - \vec{k}_c\cdot\vec{x} + \phi_c)$$

Rotating Frame Hamiltonian

$$H_D = \hbar(\omega_p - \omega_c)\left|1\right\rangle\left\langle 1\right| + \hbar\omega_p\left|2\right\rangle\left\langle 2\right|$$

$$\dot{\tilde{\rho}} = -\frac{i}{\hbar}\left[H_0 - H_D + \tilde{H}_I, \tilde{\rho}\right] + \sum_i \mathcal{D}[\tilde{L}](\tilde{\rho}) \equiv -\frac{i}{\hbar}\left[\tilde{H}, \tilde{\rho}\right] + \sum_i \mathcal{D}[\tilde{L}](\tilde{\rho})$$

$$\begin{aligned} H_0 - H_D &= \hbar\left[\omega_{10} - (\omega_p - \omega_c)\right]\left|1\right\rangle\left\langle 1\right| + \hbar(\omega_{20} - \omega_p)\left|2\right\rangle\left\langle 2\right| + \hbar\nu\left(a^\dagger a + \frac{1}{2}\right) \\ &= -\hbar\delta\left|1\right\rangle\left\langle 1\right| - \hbar\Delta\left|2\right\rangle\left\langle 2\right| + \hbar\nu\left(a^\dagger a + \frac{1}{2}\right) \end{aligned}$$

$$e^{\pm i H_D t/\hbar} = \left|0\right\rangle\left\langle 0\right| + e^{\pm i(\omega_p - \omega_c)t}\left|1\right\rangle\left\langle 1\right| + e^{\pm i\omega_p t}\left|2\right\rangle\left\langle 2\right|$$

Rotating Frame - Interaction

$$e^{\pm iH_D t/\hbar} = |0\rangle\langle 0| + e^{\pm i(\omega_p - \omega_c)t}|1\rangle\langle 1| + e^{\pm i\omega_p t}|2\rangle\langle 2|$$

$$\tilde{A} = \exp(iH_D t/\hbar) A \exp(-iH_D t/\hbar)$$

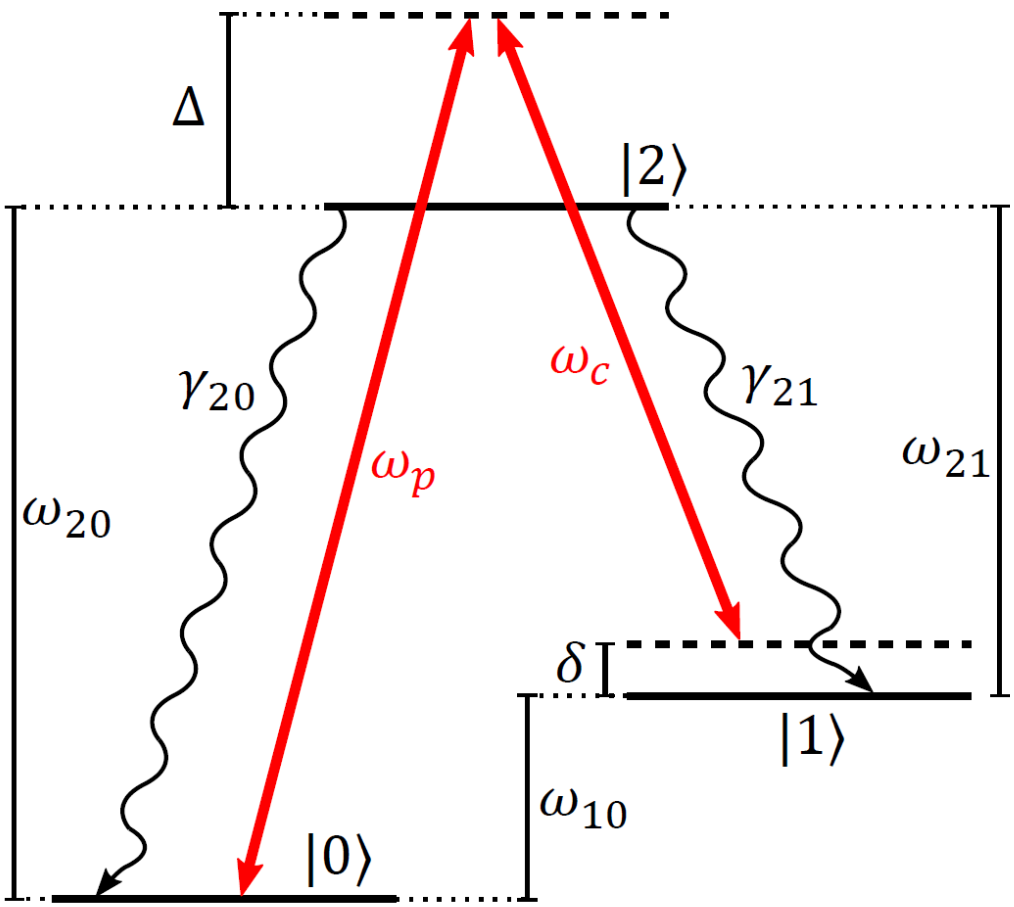
$$H_I^{(p)} = \left(\hbar\Omega_p |0\rangle\langle 2| + \hbar\Omega_p^* |2\rangle\langle 0|\right)\cos(\omega_p t - \vec{k}_p\cdot\vec{x} + \phi_p)$$

$$H_I^{(c)} = (\hbar\Omega_c |1\rangle\langle 2| + \hbar\Omega_c^* |2\rangle\langle 1|)\cos(\omega_c t - \vec{k}_c\cdot\vec{x} + \phi_c)$$

$$\tilde{H}_I^{(p)} = \left(\hbar\Omega_p e^{-i\omega_p t}|0\rangle\langle 2| + \hbar\Omega_p^* e^{i\omega_p t}|2\rangle\langle 0|\right)\cos(\omega_p t - \vec{k}_p\cdot\vec{x} + \phi_p)$$

$$\tilde{H}_I^{(c)} = \left(\hbar\Omega_c e^{-i\omega_c t}|1\rangle\langle 2| + \hbar\Omega_c^* e^{i\omega_p t}|2\rangle\langle 1|\right)\cos(\omega_c t - \vec{k}_c\cdot\vec{x} + \phi_c)$$

Relevant Frequency Scales



(Near) resonantly driven:

$$\omega_{02},\ \omega_{12},\ \omega_{01} \gg \Delta,\ \Gamma > \nu > \delta$$

$$\text{wavelength around } 600\,\text{nm}$$
$$\omega \approx 2\pi \cdot 500\,\text{THz}$$
$$\Gamma \approx 2\pi \cdot 10\,\text{MHz}$$
$$\Delta \approx \Gamma$$
$$\nu \approx 2\pi \cdot 1\,\text{MHz}$$
$$\delta \leq 2\pi \cdot 10\,\text{kHz}$$

Raman (Far off resonance):

$$\omega_{02},\ \omega_{12} \gg \Delta \gg \omega_{01} \gg \Gamma > \nu > \delta$$

$$\Delta \approx 10^5\Gamma - 10^6\Gamma$$

Rotating Wave Approximation

$$\tilde{H}_I^{(p)} = \left(\hbar\Omega_p e^{-i\omega_p t} |0\rangle \langle 2| + \hbar\Omega_p^* e^{i\omega_p t} |2\rangle \langle 0|\right) \cos(\omega_p t - \vec{k}_p \cdot \vec{x} + \phi_p)$$

$$e^{\pm i\omega t} \cos(\omega t) = \frac{1}{2}\left(1 + \cancel{e^{\pm i2\omega t}}\right)$$

$$i\hbar\dot{\rho} = H\rho - \rho H \qquad \int e^{i2\omega t} \propto 1/\omega$$

$$\tilde{H}_I^{(p)} = \frac{\hbar}{2}\left(\Omega_p e^{-i(\vec{k}_p\cdot\vec{x}-\phi_p)} |0\rangle \langle 2| + \Omega_p^* e^{i(\vec{k}_p\cdot\vec{x}-\phi_p)} |2\rangle \langle 0|\right)$$

$$\tilde{H}_I^{(c)} = \frac{\hbar}{2}\left(\Omega_c e^{-i(\vec{k}_c\cdot\vec{x}-\phi_c)} |1\rangle \langle 2| + \Omega_c^* e^{i(\vec{k}_c\cdot\vec{x}-\phi_c)} |2\rangle \langle 1|\right)$$

Motion Dependence – Lamb-Dicke Regime

$$\vec{x} = \sqrt{\hbar/2m\nu}\hat{\varepsilon}_x \left(a^\dagger + a\right)$$

$$\vec{k}_p \cdot \vec{x} = \sqrt{\frac{\hbar}{2m\nu}} k_p (\hat{k}_p \cdot \hat{\varepsilon}_x) \left(a^\dagger + a\right) \equiv \eta_p \left(a^\dagger + a\right)$$

$$\vec{k}_c \cdot \vec{x} = \sqrt{\frac{\hbar}{2m\nu}} k_c (\hat{k}_c \cdot \hat{\varepsilon}_x) \left(a^\dagger + a\right) \equiv \eta_c \left(a^\dagger + a\right)$$

Lamb-Dicke regime: $\sqrt{n}\eta < 1$

$$e^{i\eta\left(a^\dagger + a\right)} = 1 + i\eta\left(a^\dagger + a\right) + \mathcal{O}(\eta^2)$$

$$\begin{aligned}
\tilde{H}_I &= \tilde{H}_I^{(p)} + \tilde{H}_I^{(c)} \\
&= \frac{\hbar}{2}\left(\Omega_p e^{-i\eta_p(a^\dagger + a)} \left|0\right\rangle \left\langle 2\right| + \Omega_p^* e^{i\eta_p(a^\dagger + a)} \left|2\right\rangle \left\langle 0\right|\right) \\
&+ \frac{\hbar}{2}\left(\Omega_c e^{-i\eta_c(a^\dagger + a)} \left|1\right\rangle \left\langle 2\right| + \Omega_c^* e^{i\eta_c(a^\dagger + a)} \left|2\right\rangle \left\langle 1\right|\right)
\end{aligned}$$

For simplicity

$$\Omega_p \to \Omega_p e^{i\phi_p}$$

$$\Omega_c \to \Omega_c e^{i\phi_c}$$

Lindblad Dissipation Term – Rotating Frame

$$L_{20} = \sqrt{\gamma_{20}} \left|0\right\rangle \left\langle 2\right|$$
$$L_{21} = \sqrt{\gamma_{21}} \left|1\right\rangle \left\langle 2\right|$$

$$\tilde{A} = \exp(iH_D t/\hbar) A \exp(-iH_D t/\hbar)$$

$$e^{\pm iH_D t/\hbar} = \left|0\right\rangle \left\langle 0\right| + e^{\pm i(\omega_p - \omega_c)t} \left|1\right\rangle \left\langle 1\right| + e^{\pm i\omega_p t} \left|2\right\rangle \left\langle 2\right|$$

$$\tilde{L}_{20} = \sqrt{\gamma_{20}} e^{-i\omega_p t} L_{20}$$
$$\tilde{L}_{21} = \sqrt{\gamma_{21}} e^{-i\omega_c t} L_{21}$$

$$\tilde{L}^\dagger \tilde{L} = L^\dagger L \qquad \tilde{L}\rho\tilde{L}^\dagger = L\rho L^\dagger$$

$$\mathcal{D}[\tilde{L}_i](\tilde{\rho}) = \mathcal{D}[L_i](\tilde{\rho})$$

Lindblad Dissipation Term – Rotating Frame

$$L_{20} = \sqrt{\gamma_{20}}\,|0\rangle\langle 2| \qquad L_{20}^{\dagger}L_{20} = \gamma_{20}\,|2\rangle\langle 2|$$

$$L_{21} = \sqrt{\gamma_{21}}\,|1\rangle\langle 2| \qquad L_{21}^{\dagger}L_{21} = \gamma_{21}\,|2\rangle\langle 2|$$

$$\sum_{i=0}^{1}\mathcal{D}[L_{2i}](\tilde{\rho}) = (\gamma_{20}\,|0\rangle\langle 0| + \gamma_{21}\,|1\rangle\langle 1|)\,\langle 2|\tilde{\rho}|2\rangle - \frac{\gamma_{20}+\gamma_{21}}{2}\left(|2\rangle\langle 2|\,\tilde{\rho} + \tilde{\rho}\,|2\rangle\langle 2|\right)$$

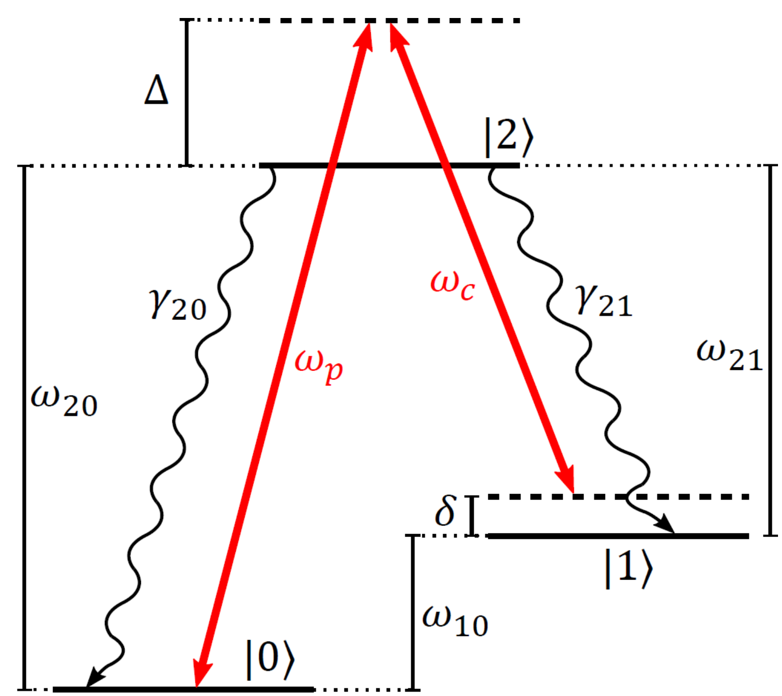
$$= \underbrace{(\gamma_{20}\,|0\rangle\langle 0| + \gamma_{21}\,|1\rangle\langle 1|)\,\tilde{\rho}_{22}}_{\text{Trace-preserving terms}} \underbrace{- \frac{\gamma_{20}+\gamma_{21}}{2}\left(|2\rangle\langle 2|\,\tilde{\rho} + \tilde{\rho}\,|2\rangle\langle 2|\right)}_{\text{Population and coherence decay terms}}$$

Three-level Hamiltonian Summary

$$\sum_{i=0}^{1} \mathcal{D}[L_{2i}](\tilde{\rho}) = \left(\gamma_{20} \left|0\right\rangle \left\langle 0\right| + \gamma_{21} \left|1\right\rangle \left\langle 1\right|\right) \tilde{\rho}_{22} - \frac{\gamma_{20} + \gamma_{21}}{2} \left(\left|2\right\rangle \left\langle 2\right| \tilde{\rho} + \tilde{\rho} \left|2\right\rangle \left\langle 2\right|\right)$$

$$\begin{aligned}\tilde{H}_I &= \tilde{H}_I^{(p)} + \tilde{H}_I^{(c)} \\ &= \frac{\hbar}{2} \left(\Omega_p e^{-i\eta_p (a^\dagger + a)} \left|0\right\rangle \left\langle 2\right| + \Omega_p^* e^{i\eta_p (a^\dagger + a)} \left|2\right\rangle \left\langle 0\right|\right) \\ &+ \frac{\hbar}{2} \left(\Omega_c e^{-i\eta_c (a^\dagger + a)} \left|1\right\rangle \left\langle 2\right| + \Omega_c^* e^{i\eta_c (a^\dagger + a)} \left|2\right\rangle \left\langle 1\right|\right)\end{aligned}$$

$$H_0 - H_D = -\hbar\delta \left|1\right\rangle \left\langle 1\right| - \hbar\Delta \left|2\right\rangle \left\langle 2\right| + \hbar\nu \left(a^\dagger a + \frac{1}{2}\right)$$



Matrix Form, No Motion (Carrier Transition)

$$\tilde{\rho}_{ij} = \langle i|\tilde{\rho}|j\rangle$$

$$\dot{\tilde{\rho}}_{00} = -\frac{i}{2}\left(\Omega_p\tilde{\rho}_{20} - \Omega_p^*\tilde{\rho}_{02}\right) + \gamma_{20}\tilde{\rho}_{22}$$
$$\dot{\tilde{\rho}}_{11} = -\frac{i}{2}\left(\Omega_c\tilde{\rho}_{21} - \Omega_c^*\tilde{\rho}_{12}\right) + \gamma_{21}\tilde{\rho}_{22}$$
$$\dot{\tilde{\rho}}_{22} = -\dot{\tilde{\rho}}_{00} - \dot{\tilde{\rho}}_{11}$$

$$\dot{\tilde{\rho}}_{01} = \dot{\tilde{\rho}}_{10}^* = -i\delta\tilde{\rho}_{01} - \frac{i}{2}\left(\Omega_p\tilde{\rho}_{21} - \Omega_c^*\tilde{\rho}_{02}\right)$$
$$\dot{\tilde{\rho}}_{02} = \dot{\tilde{\rho}}_{20}^* = -i\Delta\tilde{\rho}_{02} - \frac{i}{2}\left[\Omega_p(\tilde{\rho}_{22} - \tilde{\rho}_{00}) - \Omega_c\tilde{\rho}_{01}\right] - \frac{\gamma_{20}+\gamma_{21}}{2}\tilde{\rho}_{22}$$
$$\dot{\tilde{\rho}}_{12} = \dot{\tilde{\rho}}_{21}^* = -i(\Delta-\delta)\tilde{\rho}_{12} - \frac{i}{2}\left[\Omega_c(\tilde{\rho}_{22} - \tilde{\rho}_{11}) - \Omega_p\tilde{\rho}_{10}\right] - \frac{\gamma_{20}+\gamma_{21}}{2}\tilde{\rho}_{22}$$

Far Detuned (Raman)

$$\dot{\tilde{\rho}}_{02} = \dot{\tilde{\rho}}_{20}^{*} = -i\Delta\tilde{\rho}_{02} - \frac{i}{2}\left[\Omega_p(\tilde{\rho}_{22} - \tilde{\rho}_{00}) - \Omega_c\tilde{\rho}_{01}\right] - \frac{\gamma_{20}+\gamma_{21}}{2}\tilde{\rho}_{22}$$

$$\dot{\tilde{\rho}}_{12} = \dot{\tilde{\rho}}_{21}^{*} = -i(\Delta-\delta)\tilde{\rho}_{12} - \frac{i}{2}\left[\Omega_c(\tilde{\rho}_{22} - \tilde{\rho}_{11}) - \Omega_p\tilde{\rho}_{10}\right] - \frac{\gamma_{20}+\gamma_{21}}{2}\tilde{\rho}_{22}$$

$$\Delta \gg \Omega_c, \Omega_p, \gamma_{20}, \gamma_{21}, \delta$$

$$\begin{aligned}
\dot{\tilde{\rho}}_{02} = \dot{\tilde{\rho}}_{20}^{*} = 0 &\implies -i\Delta\tilde{\rho}_{02} + \frac{i}{2}\left[\Omega_p\tilde{\rho}_{00} + \Omega_c\tilde{\rho}_{01}\right] = 0 \\
&\implies \tilde{\rho}_{02} = \tilde{\rho}_{20}^{*} = \frac{\Omega_p}{2\Delta}\tilde{\rho}_{00} + \frac{\Omega_c}{2\Delta}\tilde{\rho}_{01} \\
\dot{\tilde{\rho}}_{12} = \dot{\tilde{\rho}}_{21}^{*} = 0 &\implies -i(\Delta-\delta)\tilde{\rho}_{12} + \frac{i}{2}\left[\Omega_c\tilde{\rho}_{11} + \Omega_p\tilde{\rho}_{10}\right] = 0 \\
&\implies \tilde{\rho}_{12} = \tilde{\rho}_{21}^{*} = \frac{\Omega_c}{2\Delta}\tilde{\rho}_{11} + \frac{\Omega_p}{2\Delta}\tilde{\rho}_{10}
\end{aligned}$$

Adiabatic Elimination

$$\dot{\tilde{\rho}}_{00} = -\frac{i}{2}\left(\Omega_p\tilde{\rho}_{20} - \Omega_p^*\tilde{\rho}_{02}\right) + \cancel{\gamma_{20}\tilde{\rho}_{22}}$$

$$\dot{\tilde{\rho}}_{11} = -\frac{i}{2}\left(\Omega_c\tilde{\rho}_{21} - \Omega_c^*\tilde{\rho}_{12}\right) + \cancel{\gamma_{21}\tilde{\rho}_{22}}$$

$$\dot{\tilde{\rho}}_{01} = \dot{\tilde{\rho}}_{10}^* = -i\delta\tilde{\rho}_{01} - \frac{i}{2}\left(\Omega_p\tilde{\rho}_{21} - \Omega_c^*\tilde{\rho}_{02}\right)$$

$$\tilde{\rho}_{02} = \tilde{\rho}_{20}^* = \frac{\Omega_p}{2\Delta}\tilde{\rho}_{00} + \frac{\Omega_c}{2\Delta}\tilde{\rho}_{01}$$

$$\tilde{\rho}_{12} = \tilde{\rho}_{21}^* = \frac{\Omega_c}{2\Delta}\tilde{\rho}_{11} + \frac{\Omega_p}{2\Delta}\tilde{\rho}_{10}$$

$$\dot{\tilde{\rho}}_{00} = i\frac{\Omega_p^*\Omega_c}{4\Delta}\tilde{\rho}_{01} - i\frac{\Omega_p\Omega_c^*}{4\Delta}\tilde{\rho}_{10}$$

$$\dot{\tilde{\rho}}_{11} = -i\frac{\Omega_p^*\Omega_c}{4\Delta}\tilde{\rho}_{01} + i\frac{\Omega_p\Omega_c^*}{4\Delta}\tilde{\rho}_{10} = -\dot{\tilde{\rho}}_{00}$$

$$\dot{\tilde{\rho}}_{01} = -i\left(\delta + \frac{|\Omega_p|^2 - |\Omega_c|^2}{4\Delta}\right)\tilde{\rho}_{01} - i\frac{\Omega_p\Omega_c^*}{4\Delta}\left(\tilde{\rho}_{11} - \tilde{\rho}_{00}\right)$$

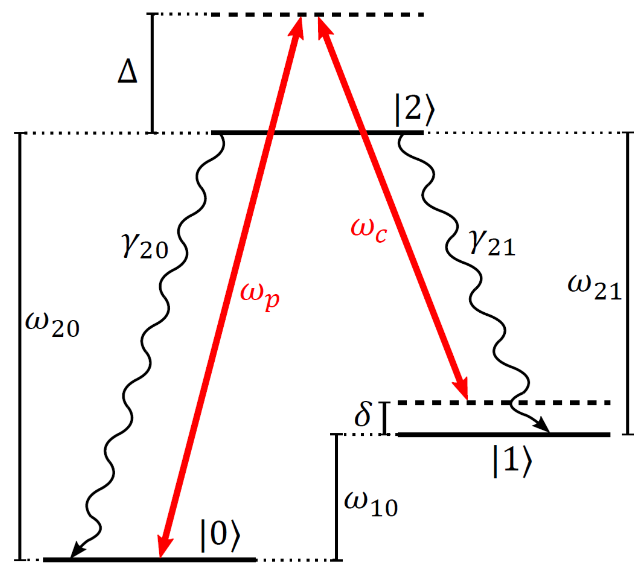
Effectively a two-level system!
Adiabatically eliminated the excited state

Effective Two-level System

$$\dot{\tilde{\rho}}_{00} = i\frac{\Omega_p^*\Omega_c}{4\Delta}\tilde{\rho}_{01} - i\frac{\Omega_p\Omega_c^*}{4\Delta}\tilde{\rho}_{10}$$

$$\dot{\tilde{\rho}}_{11} = -i\frac{\Omega_p^*\Omega_c}{4\Delta}\tilde{\rho}_{01} + i\frac{\Omega_p\Omega_c^*}{4\Delta}\tilde{\rho}_{10} = -\dot{\tilde{\rho}}_{00}$$

$$\dot{\tilde{\rho}}_{01} = -i\left(\delta + \frac{|\Omega_p|^2 - |\Omega_c|^2}{4\Delta}\right)\tilde{\rho}_{01} - i\frac{\Omega_p\Omega_c^*}{4\Delta}(\tilde{\rho}_{11} - \tilde{\rho}_{00})$$



Effective von Neumann equation $\dot{\tilde{\rho}}_{\text{eff}} = -\frac{i}{\hbar}[H_{\text{eff}}, \tilde{\rho}_{\text{eff}}]$

Effective Hamiltonian $H_{\text{eff}} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega^* & -\delta' \end{pmatrix}$

$\Omega = \frac{\Omega_p\Omega_c^*}{2\Delta}$ Effective Rabi frequency

Effective density matrix $\rho_{\text{eff}} = \begin{pmatrix} \tilde{\rho}_{00} & \tilde{\rho}_{01} \\ \tilde{\rho}_{10} & \tilde{\rho}_{11} \end{pmatrix}$

$\delta' = \delta + \frac{|\Omega_p|^2 - |\Omega_c|^2}{4\Delta}$ Effective detuning

Noteworthy Details

$$\delta' = \delta + \frac{|\Omega_p|^2 - |\Omega_c|^2}{4\Delta}$$

Differential AC Stark shift

Differential phase shift

$$\Omega = \frac{\Omega_p \Omega_c^*}{2\Delta} \propto E_0^{(p)} E_0^{(c)*} e^{i(\phi_p - \phi_c)}$$

Scaling: $1/\Delta$

Spontaneous emission rate scaling: $1/\Delta^2$

For good Raman transitions:
- High laser intensity
- Large detuning