Magnetic Moment and L-S Coupling

January 29, 2020

Problem 1: Consider the evolution of a magnetic moment $\vec{\mu}$ subjected to a combined oscillating and static magnetic field $\vec{B} = -B_1[\cos(\omega t)\hat{e}_x + \sin(\omega t)\hat{e}_y] - B_0\hat{e}_z$.

(a) Show that the Hamiltonian $H = -\vec{\mu} \cdot \vec{B}$ for this system can be written as:

$$H = \frac{\hbar}{2} \begin{pmatrix} \omega_L & \omega_R e^{-i\omega t} \\ \omega_R e^{i\omega t} & -\omega_L \end{pmatrix} \tag{1}$$

where $\omega_L = \gamma B_0$ and $\omega_R = \gamma B_1$ are the Larmor and Rabi frequencies associated with the static and rotating fields, resp., and γ is the gyromagnetic ratio.

(b) Using the basis states $|e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and defining a general state $|\psi\rangle = c_g(t)\,|g\rangle + c_e(t)\,|e\rangle$, write down and solve the equations of motion in terms of the initial conditions $c_g(0)$ and $c_e(0)$.

(c) Use the `qutip` module to write a program to solve the Schrödinger equation for this Hamiltonian. Plot the probability of finding the electron in $|e\rangle$ as a function of time for the parameters $(\omega_L, \omega_R, \omega) = (1, 0.5, 1.75)$, and assuming the initial state is $|g\rangle$.

(d) QuTiP has a handy tool for visualizing states on the Bloch sphere. Create an animation where you plot the state vector on the Bloch sphere at each point in time. You can do this by saving your state vectors in a sequence of `*.png` files, and using a separate utility to combine them into a video file (for example, using `ffmpeg`). Create an animation for $(\omega_L, \omega_R, \omega) = (1, 0, 0)$ and starting in an equal superposition of $|g\rangle$ and $|e\rangle$. Describe the observed behavior.

(e) Create another animation, this time with the parameters $(\omega_L, \omega_R, \omega) = (\pi, 1, \pi)$, and starting in $|g\rangle$. Describe the observed behavior.

(f) Transform the Hamiltonian into a frame where it no longer explicitly depends on time.

(g) Create an animation for the state vector solutions to the effective Hamiltonian you derived in (f) using the same parameters as in (e). How does the simulation differ? Describe any differences and similarities. Describe what we have effectively done.

Problem 2: It is possible to change the basis states from uncoupled $|m_l\, m_s\rangle$ basis states to coupled $|j\, m_j\rangle$ basis states by using the following equation:

$$
\begin{aligned}
|j\, m_j\rangle &= \sum_{m_l=-l}^{l} \sum_{m_s=-s}^{s} \langle l\, m_l\, s\, m_s | j\, m_j\rangle\, |m_l\, m_s\rangle \\
&= \sum_{m_l=-l}^{l} \sum_{m_s=-s}^{s} C^{j\, m_j}_{l\, m_l\, s\, m_s}\, |m_l\, m_s\rangle
\end{aligned}
\tag{2}
$$

Here the $\langle l\, m_l\, s\, m_s | j\, m_j\rangle$ terms are known as the Clebsch-Gordan (CG) coefficients. There exists a general solution for the CG coefficients, and these solutions have been tabulated and are widely available. Nevertheless, it is important to know where they come from and how they are obtained. We will derive the CG coefficients for the

$$
{}^2D_{3/2} \quad m_j = 1/2 \tag{3}
$$

state in two ways. The overall sign for any set of states is arbitrarily chosen, so feel free to ignore it.

(a) Write $\hat{\boldsymbol{J}}^2 - \hat{\boldsymbol{L}}^2 - \hat{\boldsymbol{S}}^2$ in terms of z-components of $\hat{\boldsymbol{L}}$ and $\hat{\boldsymbol{S}}$, and the ladder operators $\hat{L}_\pm$ and $\hat{S}_\pm$. Use this result, together with Eq. 2, to obtain the requested CG coefficients.

(b) Start from what is called the "stretched state", i.e. the state with maximal (or minimal) j and m_j for a given l and s. In this case the upper $(+)$ and lower $(-)$ stretched states are $j = 5/2, m_j = \pm 5/2$. Use the ladder operators $J_\pm$ and orthogonality conditions to obtain the desired CG coefficients.

(c) Confirm your answers using the QuTiP module's built-in functions.