Entanglement of ions using Molmer-Sorensen Gates

ECE 590.01

Quantum Engineering with Atoms

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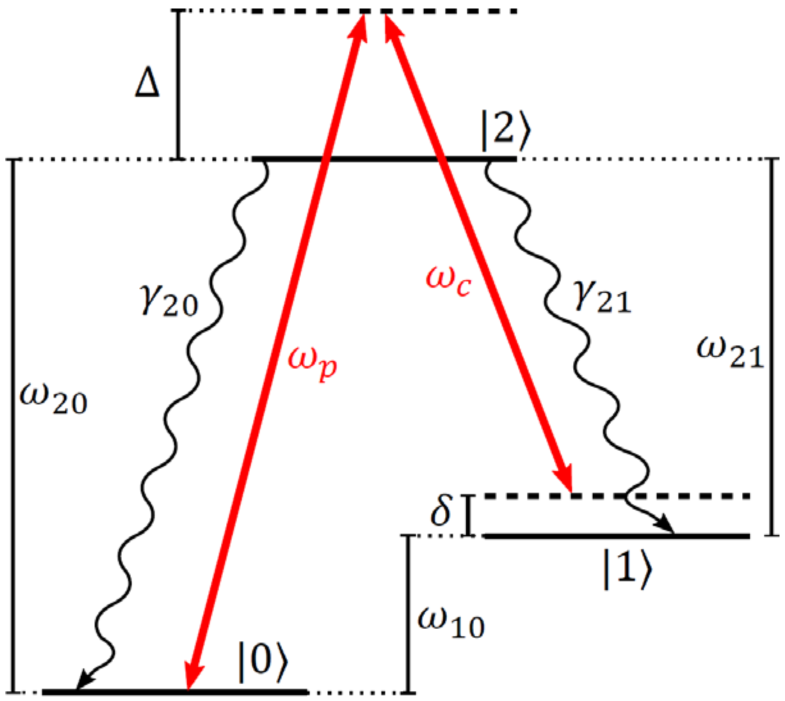
Effective Two-level System

$$\dot{\tilde{\rho}}_{00} = i\frac{\Omega_p^*\Omega_c}{4\Delta}\tilde{\rho}_{01} - i\frac{\Omega_p\Omega_c^*}{4\Delta}\tilde{\rho}_{10}$$

$$\dot{\tilde{\rho}}_{11} = -i\frac{\Omega_p^*\Omega_c}{4\Delta}\tilde{\rho}_{01} + i\frac{\Omega_p\Omega_c^*}{4\Delta}\tilde{\rho}_{10} = -\dot{\tilde{\rho}}_{00}$$

$$\dot{\tilde{\rho}}_{01} = -i\left(\delta + \frac{|\Omega_p|^2 - |\Omega_c|^2}{4\Delta}\right)\tilde{\rho}_{01} - i\frac{\Omega_p\Omega_c^*}{4\Delta}(\tilde{\rho}_{11} - \tilde{\rho}_{00})$$



Effective von Neumann equation $$\dot{\tilde{\rho}}_{\text{eff}} = -\frac{i}{\hbar}[H_{\text{eff}}, \tilde{\rho}_{\text{eff}}]$$

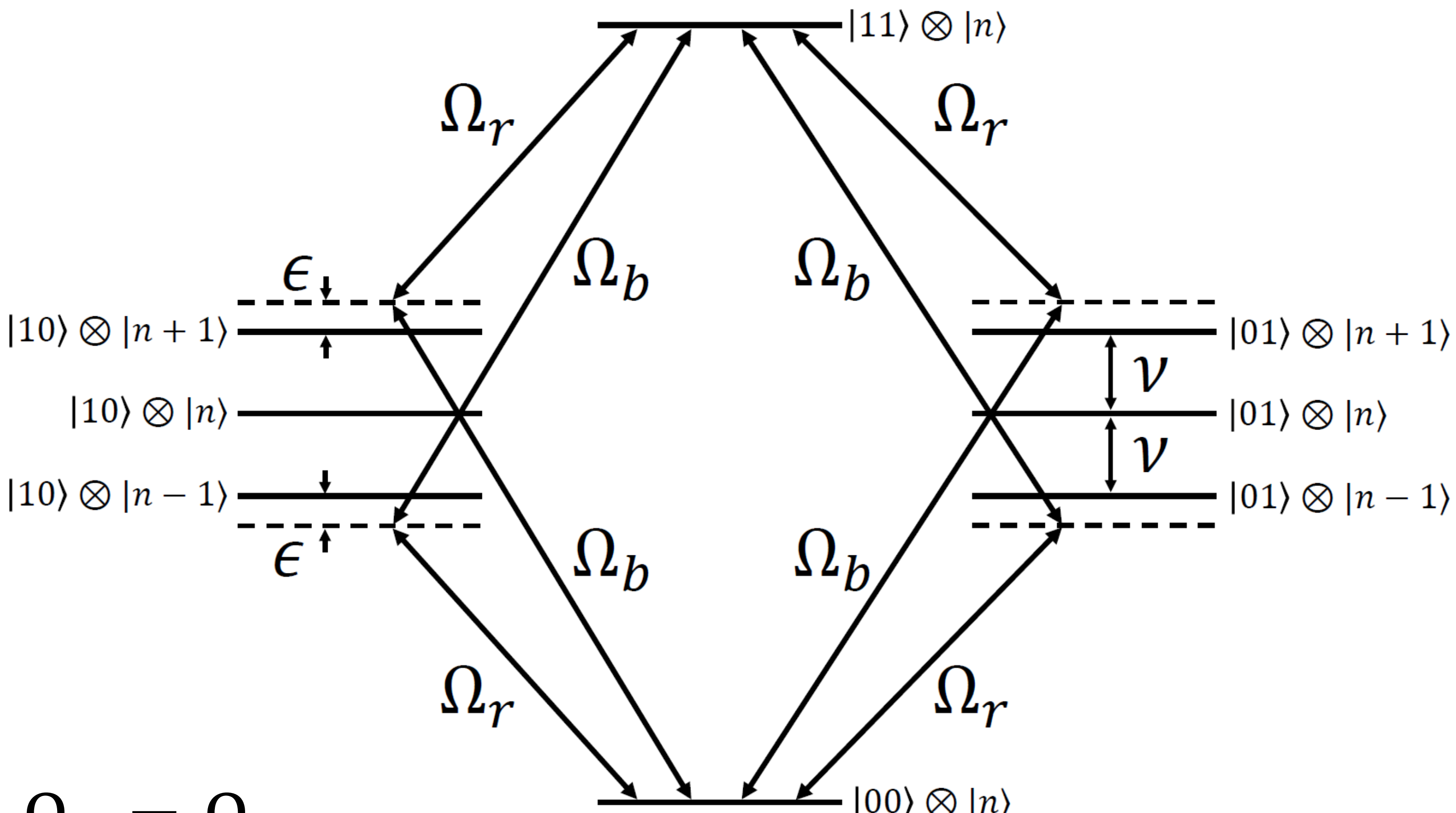
Effective Hamiltonian $$H_{\text{eff}} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega^* & -2\delta' \end{pmatrix}$$

$$\Omega = \frac{\Omega_p\Omega_c^*}{2\Delta}$$ Effective Rabi frequency

Effective density matrix $$\rho_{\text{eff}} = \begin{pmatrix} \tilde{\rho}_{00} & \tilde{\rho}_{01} \\ \tilde{\rho}_{10} & \tilde{\rho}_{11} \end{pmatrix}$$

$$\delta' = \delta + \frac{|\Omega_p|^2 - |\Omega_c|^2}{4\Delta}$$ Effective detuning

Molmer-Sorensen Gate – Interfering Paths



$$\Omega_r = \Omega_b = \Omega$$

$$\Omega_{\text{upper}} = -(n+1)\frac{\eta^2\Omega^2}{\epsilon}$$

$$\Omega_{\text{lower}} = n\frac{\eta^2\Omega^2}{\epsilon}$$

$$|\Omega_{\text{eff}}| = |\Omega_{\text{upper}} + \Omega_{\text{lower}}| = \frac{\eta^2\Omega^2}{\epsilon}$$

Adding the Motion Back

$$\delta' = \delta + (|\Omega_p|^2 - \Omega_c|^2)/4\Delta$$

$$H_{\mathrm{eff}} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega^* & -2\delta' \end{pmatrix}$$

$$\Omega = \Omega_p \Omega_c^* / 2\Delta$$

$$\Omega \rightarrow \Omega e^{-i\eta(a^\dagger + a)} \qquad \eta = \eta_c - \eta_p$$

$$H_{\mathrm{I,eff}} = -\hbar\delta' \left|1\right\rangle \left\langle 1\right| + \hbar\Omega\hat{\sigma}_- e^{i\eta(a^\dagger + a)} + \hbar\Omega^*\hat{\sigma}_+ e^{-i\eta(a^\dagger + a)}$$

$$H_{\mathrm{eff}} = H_{\mathrm{I,eff}} + \hbar\nu\left(a^\dagger a + \frac{1}{2}\right)$$

Interaction Picture: Atomic Operators

$$H_{\mathrm{I,eff}} = -\hbar\delta' \left|1\right\rangle \left\langle 1\right| + \hbar\Omega\hat{\sigma}_- e^{i\eta(a^\dagger+a)} + \hbar\Omega^*\hat{\sigma}_+ e^{-i\eta(a^\dagger+a)}$$

$$H_{\mathrm{eff}} = H_{\mathrm{I,eff}} + \hbar\nu\left(a^\dagger a + \frac{1}{2}\right)$$

$$H_D = -\hbar\delta' \left|1\right\rangle \left\langle 1\right| + \hbar\nu(a^\dagger a + 1/2)$$

$$\tilde{A} = \exp(iH_D t/\hbar) A \exp(-iH_D t/\hbar)$$

$$e^{iH_D/\hbar t}\hat{\sigma}_- e^{-iH_D/\hbar t} = e^{i\delta' t}\hat{\sigma}_-$$

To the Interaction Picture – Motional Operators

$$H_{\mathrm{I,eff}} = -\hbar\delta' \left|1\right\rangle \left\langle 1\right| + \hbar\Omega\hat{\sigma}_{-}e^{i\eta(a^\dagger+a)} + \hbar\Omega^*\hat{\sigma}_{+}e^{-i\eta(a^\dagger+a)}$$

$$H_{\mathrm{eff}} = H_{\mathrm{I,eff}} + \hbar\nu\left(a^\dagger a + \frac{1}{2}\right)$$

$$H_D = -\hbar\delta' \left|1\right\rangle \left\langle 1\right| + \hbar\nu(a^\dagger a + 1/2)$$

$$e^{iH_D/\hbar t}ae^{-iH_D/\hbar t} = e^{i\nu t a^\dagger a}ae^{-i\nu t a^\dagger a}$$

$$\begin{aligned} a(a^\dagger a)^n &= aa^\dagger a(a^\dagger a)^{n-1} \\ &= (a^\dagger a + 1)a(a^\dagger a)^{n-1} \\ &= (a^\dagger a + 1)^n a \end{aligned}$$

To the Interaction Picture – Motional Operators

$$a(a^\dagger a)^n = (a^\dagger a + 1)^n a$$

$$\begin{aligned}
e^{iH_D/\hbar t} a e^{-iH_D/\hbar t} &= e^{i\nu t a^\dagger a} a e^{-i\nu t a^\dagger a} \\
&= e^{i\nu t a^\dagger a} \sum_n \frac{(-i\nu t)^n}{n!} a(a^\dagger a)^n \\
&= e^{i\nu t a^\dagger a} \sum_n \frac{(-i\nu t)^n}{n!} (a^\dagger a + 1)^n a \\
&= e^{i\nu t a^\dagger a} e^{-i\nu t (a^\dagger a + 1)} a \\
&= e^{-i\nu t} a \qquad\qquad a \to a e^{-i\nu t}
\end{aligned}$$

Interaction Picture Hamiltonian

$$H_{\mathrm{I,eff}} = -\hbar\delta' \left|1\right\rangle \left\langle 1\right| + \hbar\Omega\hat{\sigma}_{-}e^{i\eta(a^{\dagger}+a)} + \hbar\Omega^{*}\hat{\sigma}_{+}e^{-i\eta(a^{\dagger}+a)}$$

$$H_{\mathrm{eff}} = H_{\mathrm{I,eff}} + \hbar\nu\left(a^{\dagger}a + \frac{1}{2}\right)$$

$$H_D = -\hbar\delta' \left|1\right\rangle \left\langle 1\right| + \hbar\nu(a^{\dagger}a + 1/2)$$

$$\tilde{A} = \exp(iH_D t/\hbar) A \exp(-iH_D t/\hbar)$$

$$\dot{\tilde{\rho}} = -\frac{i}{\hbar}\left[H_0 - H_D + \tilde{H}_I, \tilde{\rho}\right]$$

$$\tilde{H}_{\mathrm{eff}} = \frac{\hbar}{2}\left[\Omega e^{i\delta' t}\hat{\sigma}_{-}e^{i\eta(a^{\dagger}e^{i\nu t}+ae^{-i\nu t})} + \Omega^{*}e^{-i\delta' t}\hat{\sigma}_{+}e^{-i\eta(a^{\dagger}e^{i\nu t}+ae^{-i\nu t})}\right]$$

Matched Fields

$$\begin{aligned}\tilde{H}_{\text{eff}} &= \tilde{H}_{\text{eff}}^{(1)} + \tilde{H}_{\text{eff}}^{(2)} \\ &= \frac{\hbar}{2}\Big[\Omega_1 e^{i\delta_1' t}\hat{\sigma}_-^{(1)} e^{i\eta_1(a^\dagger e^{i\nu t}+ae^{-i\nu t})} + \Omega_1^* e^{-i\delta_1' t}\hat{\sigma}_+^{(1)} e^{-i\eta_1(a^\dagger e^{i\nu t}+ae^{-i\nu t})} \\ &\quad +\Omega_2 e^{i\delta_2' t}\hat{\sigma}_-^{(2)} e^{i\eta_2(a^\dagger e^{i\nu t}+ae^{-i\nu t})} + \Omega_2^* e^{-i\delta_2' t}\hat{\sigma}_+^{(2)} e^{-i\eta_2(a^\dagger e^{i\nu t}+ae^{-i\nu t})}\Big]\end{aligned}$$

$$\Omega_1 = \Omega_2 = \Omega,\ \eta_1 = \eta_2 = \eta,\ \text{and}\ \delta_1' = \delta_2' = \delta'$$

$$\tilde{H}_{\text{eff}} = \frac{\hbar}{2}\Big[\Omega e^{i\delta' t}\hat{S}_- e^{i\eta(a^\dagger e^{i\nu t}+ae^{-i\nu t})} + \Omega^* e^{-i\delta' t}\hat{S}_+ e^{-i\eta(a^\dagger e^{i\nu t}+ae^{-i\nu t})}\Big]$$

$$\hat{S}_- = \hat{\sigma}_-^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{\sigma}_-^{(2)}$$

Lamb-Dicke Regime

$$\tilde{H}_{\text{eff}} = \frac{\hbar}{2}\left[\Omega e^{i\delta' t}\hat{S}_{-}e^{i\eta(a^{\dagger}e^{i\nu t}+ae^{-i\nu t})} + \Omega^{*}e^{-i\delta' t}\hat{S}_{+}e^{-i\eta(a^{\dagger}e^{i\nu t}+ae^{-i\nu t})}\right]$$

$$\eta\sqrt{\bar{n}} \leq 1$$

$$\begin{aligned}\tilde{H}_{\text{eff}} \approx \frac{\hbar}{2}\Big[& \boxed{\Omega e^{i\delta' t}\hat{S}_{-} + \Omega e^{-i\delta' t}\hat{S}_{+}} \\ & + \boxed{i\eta\left(\Omega e^{i(\delta-\nu)}\hat{S}_{-}a - \Omega^{*}e^{-i(\delta-\nu)}\hat{S}_{+}a^{\dagger}\right)} \\ & + \boxed{i\eta\left(\Omega e^{i(\delta+\nu)}\hat{S}_{-}a^{\dagger} - \Omega^{*}e^{-i(\delta+\nu)}\hat{S}_{+}a\right)}\Big]\end{aligned}$$

Carrier

Blue sideband

Red sideband

Multiple Beam Pairs + Rotating Wave Approximations

$$\tilde{H}_{\text{eff}} \approx \frac{\hbar}{2}\left[\Omega e^{i\delta' t}\hat{S}_- + \Omega e^{-i\delta' t}\hat{S}_+\right.$$
$$+ \boxed{i\eta\left(\Omega e^{i(\delta-\nu)}\hat{S}_- a - \Omega^* e^{-i(\delta-\nu)}\hat{S}_+ a^\dagger\right)}$$
$$\left.+ \boxed{i\eta\left(\Omega e^{i(\delta+\nu)}\hat{S}_- a^\dagger - \Omega^* e^{-i(\delta+\nu)}\hat{S}_+ a\right)}\right]$$

Slower compared to carrier

$$\delta' = \nu + \delta_b \ll \nu \xrightarrow{\text{RWA}} H_B = \frac{\hbar}{2} i\eta\left(\Omega_b e^{i\delta_b t}\hat{S}_- a - \Omega_b^* e^{-i\delta_b t}\hat{S}_+ a^\dagger\right)$$

$$\delta' = -\nu + \delta_r \ll \nu \xrightarrow{\text{RWA}} H_R = \frac{\hbar}{2} i\eta\left(\Omega_r e^{i\delta_r t}\hat{S}_- a^\dagger - \Omega_r^* e^{-i\delta_r t}\hat{S}_+ a\right)$$

Molmer-Sorensen Hamiltonian

$$H_B = \frac{\hbar}{2} i\eta \left(\Omega_b e^{i\delta_b t}\hat{S}_- a - \Omega_b^* e^{-i\delta_b t}\hat{S}_+ a^\dagger\right)$$

$$H_R = \frac{\hbar}{2} i\eta \left(\Omega_r e^{i\delta_r t}\hat{S}_- a^\dagger - \Omega_r^* e^{-i\delta_r t}\hat{S}_+ a\right)$$

$$\Omega_r = \Omega_0 e^{i\phi_r}$$

$$\Omega_b = \Omega_0 e^{i\phi_b}$$

$$\delta_b = -\delta_r = \epsilon$$

$$\begin{aligned} H_{MS} &= i\frac{\hbar\eta\Omega_0}{2}\left[e^{i(\epsilon t+\phi_b)}\hat{S}_- a - e^{-i(\epsilon t+\phi_b)}\hat{S}_+ a^\dagger + e^{-i(\epsilon t-\phi_r)}\hat{S}_- a^\dagger - e^{i(\epsilon t-\phi_r)}\hat{S}_+ a\right] \\ &= i\frac{\hbar\eta\Omega_0}{2}\left[(\hat{S}_- e^{i\phi_s} - \hat{S}_+ e^{-i\phi_s})(a e^{i\epsilon t} e^{i\phi_m} + a^\dagger e^{-i\epsilon t} e^{-i\phi_m})\right] \\ &\equiv i\frac{\hbar\eta\Omega_0}{2}\hat{S}\otimes\hat{A}(t) \end{aligned}$$

$$2\phi_s = \phi_b + \phi_r$$

$$2\phi_m = \phi_b - \phi_r$$

$$\hat{S} = \hat{S}_- e^{i\phi_s} - \hat{S}_+ e^{-i\phi_s}$$

Time Evolution Operator

Time-independent: $U(t) = \exp\left(-\frac{i}{\hbar}Ht\right)$

Time-dependent: $U(t) = \exp\left[\sum_{k=1}^{\infty} M_k(t)\right]$

Magnus Expansion:

$$M_1(t) = -\frac{i}{\hbar}\int_0^t H(t_1)dt_1$$

$$M_2(t) = \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 \int_0^t \int_0^{t_1} [H(t_1), H(t_2]\, dt_2 dt_1$$

$$M_3(t) = \frac{1}{6}\left(-\frac{i}{\hbar}\right)^3 \int_0^t \int_0^{t_1} \int_0^{t_2} [H(t_1), [H(t_2), H(t_3]] + [H(t_3), [H(t_2), H(t_1]]\, dt_3 dt_2 dt_1$$

$\vdots \qquad \vdots$

First-Order Term

$$M_1(t) = -\frac{i}{\hbar}\int_0^t H(t_1)dt_1 \qquad H_{MS} = i\frac{\hbar\eta\Omega_0}{2}\hat{S}\otimes\hat{A}(t)$$

$$\begin{aligned} M_1(t) &= \frac{\eta\Omega_0}{2}\hat{S}\int_0^t (ae^{i\epsilon t_1}e^{i\phi_m} + a^\dagger e^{-i\epsilon t_1}e^{-i\phi_m})dt_1 \\ &= \frac{\eta\Omega_0}{2}\hat{S}\left(a\frac{e^{i\epsilon t}-1}{i\epsilon}e^{i\phi_m} - a^\dagger\frac{e^{-i\epsilon t}-1}{i\epsilon}e^{-i\phi_m}\right) \\ &= \hat{S}\left[\alpha(t)a + \alpha^*(t)a^\dagger\right] \end{aligned}$$

$$\alpha(t) = \frac{\eta\Omega_0}{2}\frac{e^{i\epsilon t}-1}{i\epsilon}e^{i\phi_m} = \frac{\eta\Omega_0}{\epsilon}e^{i\epsilon t/2}\sin(\epsilon t/2)e^{i\phi_m}$$

Second-Order Term

$$M_2(t) = \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 \int_0^t \int_0^{t_1} [H(t_1), H(t_2]\, dt_2 dt_1$$

$$H_{MS} = \; i\frac{\hbar\eta\Omega_0}{2}\hat{S}\otimes\hat{A}(t)$$

$$\begin{aligned}\left[\hat{S}\otimes\hat{A}(t_1), \hat{S}\otimes\hat{A}(t_2)\right] &= \hat{S}^2\hat{A}(t_1)\hat{A}(t_2) - \hat{S}^2\hat{A}(t_2)\hat{A}(t_1) \\ &= \hat{S}^2\left[\hat{A}(t_1), \hat{A}(t_2)\right] \\ &= \hat{S}^2\left[ae^{i\epsilon t_1}e^{i\phi_m} + a^\dagger e^{-i\epsilon t_1}e^{-i\phi_m}, ae^{i\epsilon t_2}e^{i\phi_m} + a^\dagger e^{-i\epsilon t_2}e^{-i\phi_m}\right] \\ &= \hat{S}^2\left([a, a^\dagger]e^{i\epsilon(t_1-t_2)} + [a^\dagger, a]e^{i\epsilon(t_2-t_1)}\right) \\ &= \hat{S}^2 2i\sin\left[\epsilon(t_1 - t_2)\right]\end{aligned}$$

Second-Order Term

$$M_2(t) = \frac{1}{2}\left(-\frac{i}{\hbar}\right)^2 \int_0^t \int_0^{t_1} [H(t_1), H(t_2]\, dt_2 dt_1$$

$$\begin{aligned} M_2(t) &= \frac{1}{2}\frac{\eta^2\Omega_0^2}{4} \int_0^t \int_0^{t_1} [S \otimes A(t_1), S \otimes A(t_2)]\, dt_2 dt_1 \\ &= i\frac{\eta^2\Omega_0^2}{4}\hat{S}^2 \int_0^t \frac{1}{\epsilon}[1 - \cos(\epsilon t_1)]\, dt_1 \\ &= i\frac{\eta^2\Omega_0^2}{4\epsilon}\hat{S}^2 \left[t - \frac{\sin(\epsilon t)}{\epsilon}\right] \\ &= i\hat{S}^2\Phi(t) \end{aligned}$$

$$\Phi(t) = \left(\frac{\eta\Omega_0}{2\epsilon}\right)^2 [\epsilon t - \sin(\epsilon t)]$$

$$[M_2(t_1), H(t_2)] = 0$$

Therefore higher order terms in Magnus expansion vanish

Molmer-Sorensen Gate – Time Evolution

Spin-motion coupling   Spin-dependent phase shift

$$U_{MS}(t) = \exp\left[\boxed{\hat{S}\left(\alpha(t)a + \alpha^*(t)a^\dagger\right)} + \boxed{i\hat{S}^2\Phi(t)}\right]$$

$$\alpha(t) = \frac{\eta\Omega_0}{2}\frac{e^{i\epsilon t}-1}{i\epsilon}e^{i\phi_m} = \frac{\eta\Omega_0}{\epsilon}e^{i\epsilon t/2}\sin(\epsilon t/2)e^{i\phi_m}$$

$$\Phi(t) = \left(\frac{\eta\Omega_0}{2\epsilon}\right)^2\left[\epsilon t - \sin(\epsilon t)\right]$$

$$\alpha(t_g) = 0 \longrightarrow \epsilon t_g = 2\pi \longrightarrow \Phi_g = \frac{\pi}{2}\left(\frac{\eta\Omega_0}{\epsilon}\right)^2$$

$$U_{MS}(t_g) = e^{i\boxed{\hat{S}^2}\Phi_g}$$

Effect on atomic state?

Spin Operator

$$\hat{S}^2 = (\hat{S}_- e^{i\phi_s} - \hat{S}_+ e^{-i\phi_s})^2$$
$$= \hat{S}_-^2 e^{2i\phi_s} + \hat{S}_+^2 e^{-2i\phi_s} - \hat{S}_+\hat{S}_- - \hat{S}_-\hat{S}_+$$

$$\hat{S}_-^2 = (\hat{\sigma}_-^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{\sigma}_-^{(2)})(\hat{\sigma}_-^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{\sigma}_-^{(2)})$$
$$= 2\hat{\sigma}_-^{(1)} \otimes \hat{\sigma}_-^{(2)}$$

$$\hat{S}_+\hat{S}_- = (\hat{\sigma}_+^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{\sigma}_+^{(2)})(\hat{\sigma}_-^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{\sigma}_-^{(2)})$$
$$= \hat{\sigma}_+^{(1)}\hat{\sigma}_-^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \hat{\sigma}_+^{(2)}\hat{\sigma}_-^{(2)} + \hat{\sigma}_+^{(1)} \otimes \hat{\sigma}_-^{(2)} + \hat{\sigma}_-^{(1)} \otimes \hat{\sigma}_+^{(2)}$$

$$\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+ = 2(\hat{I}^{(1)} \otimes \hat{I}^{(2)} + \hat{\sigma}_+^{(1)} \otimes \hat{\sigma}_-^{(2)} + \hat{\sigma}_-^{(1)} \otimes \hat{\sigma}_+^{(2)})$$

$$\hat{\sigma}_+\hat{\sigma}_- = |1\rangle\langle 1| \qquad \hat{\sigma}_-\hat{\sigma}_+ = |0\rangle\langle 0| \qquad \hat{\sigma}_+\hat{\sigma}_- - \hat{\sigma}_-\hat{\sigma}_+ = \hat{I}$$

Matrix Form – Eigenvalues + Eigenvectors

$$\hat{S}^2 = 2\left(e^{2i\phi_s}\hat{\sigma}_-^{(1)} \otimes \hat{\sigma}_-^{(2)} + e^{-2i\phi_s}\hat{\sigma}_+^{(1)} \otimes \hat{\sigma}_+^{(2)} - \hat{\sigma}_+^{(1)} \otimes \hat{\sigma}_-^{(2)} + \hat{\sigma}_-^{(1)} \otimes \hat{\sigma}_+^{(2)} - \hat{I}^{(1)} \otimes \hat{I}^{(2)}\right)$$

$$= \begin{pmatrix} -2 & 0 & 0 & 2e^{2i\phi_s} \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & 0 \\ 2e^{-2i\phi_s} & 0 & 0 & -2 \end{pmatrix}$$

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda_1 = \lambda_2 = -4 : \qquad |v_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad |v_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -e^{-2i\phi_s} \end{pmatrix}$$

$$\lambda_3 = \lambda_4 = 0 : \qquad |v_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad |v_4\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ e^{-2i\phi_s} \end{pmatrix}$$

Back to Bra-Ket notation

$$\lambda_1 = \lambda_2 = -4 \qquad |v_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \qquad |v_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \qquad \lambda_3 = \lambda_4 = 0$$

$$|v_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{-2i\phi_s}|11\rangle) \qquad |v_4\rangle = \frac{1}{\sqrt{2}}(|00\rangle - e^{-2i\phi_s}|11\rangle)$$

$$U_{MS}(t_g) = e^{i\hat{S}^2\Phi_g} \qquad |00\rangle = (|v_2\rangle + |v_4\rangle)/\sqrt{2}.$$

$$\begin{aligned}
U_{MS}(t_g)|00\rangle &= e^{i\hat{S}^2\Phi_g}\frac{1}{\sqrt{2}}(|v_2\rangle + |v_4\rangle) \\
&= \frac{1}{\sqrt{2}}\left(e^{-4i\Phi_g}|v_2\rangle + |v_4\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left[\left(1 + e^{-4i\Phi_g}\right)|00\rangle + \left(-1 + e^{-4i\Phi_g}\right)e^{-2i\phi_s}|11\rangle\right] \\
&= e^{-2i\Phi_g}\left[\cos(2\Phi_g)|00\rangle - i\sin(2\Phi_g)e^{-2i\phi_s}|11\rangle\right]
\end{aligned}$$

Entanglement

$$U_{MS}(t_g)\left|00\right\rangle = e^{-2i\Phi_g}\left[\cos(2\Phi_g)\left|00\right\rangle - i\sin(2\Phi_g)e^{-2i\phi_s}\left|11\right\rangle\right]$$

$$2\Phi_g = \pi/4$$

$$U_{MS}(t_g)\left|00\right\rangle = \frac{e^{-i\pi/4}}{\sqrt{2}}\left(\left|00\right\rangle - ie^{-2i\phi_s}\left|11\right\rangle\right)$$

$$2\Phi_g = \pi\left(\frac{\eta\Omega_0}{\epsilon}\right)^2 = \frac{\pi}{4}$$

$$\epsilon = 2\eta\Omega_0 \qquad t_g = \frac{2\pi}{\epsilon} = \frac{\pi}{\eta\Omega}$$