ECE 590.01 Quantum Engineering with Atoms

Spring 2020
Final Exam
Due on Sunday 4/26/2020 10:00 pm EDT

Instructor: Geert Vrijsen and Jungsang Kim

Name: ____

I hereby certify that I worked on this mid-term exam abiding by the rules set forth by the instructor, as indicated below. I understand that the violation of these rules will be considered an academic misconduct, and will be subject to punishment. I also certify that I participated in the exam following the academic integrity and honesty anticipated for all Duke students.

Exam Rules:

1. The exam is an open book, open source test, simulating a research situation. The students are allowed to use any references one can locate, from lecture notes, any references (books or papers), or internet searches.
2. The students are allowed to discuss with and share ideas with other students enrolled in the class. However, the students are not allowed to discuss the problems with other members of the Duke community, or other members of the research community in general, including the instructors.
3. Any code used to solve the problems are written by myself, although discussions with other students in the class and consultations of references are allowed.

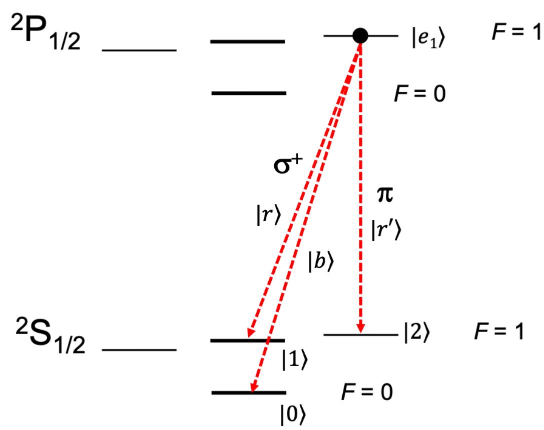
Signature ____ Date___________

1. **Explain the following concepts:**
   a. What is a coherent dark state in a three-level, lambda(λ)-system, and how is it formed? (5 points)

   b. What is the process of adiabatic elimination in considering a far-detuned Raman transition in a three-level, lambda(λ)-system, and under what conditions is this approximation justified? (5 points)

2. **Entanglement Swapping:** Consider two maximally entangled states, $|\psi_+\rangle_{12} = (|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2)/\sqrt{2}$ between qubits #1 and #2, and $|\psi_+\rangle_{34} = (|0\rangle_3|0\rangle_4 + |1\rangle_3|1\rangle_4)/\sqrt{2}$ between qubits #3 and #4. Following the procedure discussed in class, describe the procedure for teleporting the state of qubit #2 to qubit #4, by performing a Bell-state measurement between qubits #2 and #3. Show that this results in a maximally entangled state between qubits #1 and #4. (10 points)

3. **Orthogonality between σ and π emission:** Selection rules for dipole transition (Lecture Notes 7.1.4) are given by $\Delta l = \pm 1$ and $\Delta m = 0, \pm 1$. When $\Delta m = \pm 1$, the polarization of the emitted photon is $\sigma_{\pm}$, and is called the σ- emission. When $\Delta m = 0$, the polarization of the emitted photon is π, and is called the π - emission. These two polarization components are orthogonal, and it should be possible to filter one polarization against the other. In the problems below, we choose z-axis as the quantization axis, and consider practical methods for collecting the emitted photons of one type of polarization and not the other. We further consider dipole emission between a S ground state orbital ($l_f = 0$) and a P excited state orbital ($l_i = 1$) for a hydrogen-like atom with a nuclear spin of $I = ½$, so that $\Delta l = 1$.



a. Consider collecting the emitted photons along the z-axis. We want to increase the solid angle of the collection cone, so that a large fraction of the emitted photons can be collected. Show that in spherical coordinates, the electric field corresponding to the dipole radiation takes the form (10 points):

$$\vec{E}_{\pi} = \frac{ie^{ikr}}{r}\sqrt{\frac{3}{8\pi}}\sin\theta\,\hat{\theta}$$

$$\vec{E}_{\sigma_{\pm}} = \frac{ie^{ikr}}{r}e^{\pm i\phi}\sqrt{\frac{3}{16\pi}}\left(\pm\cos\theta\,\hat{\theta} + i\hat{\phi}\right)$$

[Hint: See T. Kim, P. Maunz and J. Kim, Phys. Rev. A84, 063423 (2011).]

b. Imagine a situation where we are collecting photons into a cylindrically symmetric mode, such as into a single mode fiber. This can be described in cylindrical coordinates along the z axis. We can express the electric field in a single mode fiber by the Gaussian mode function $\vec{G}(\rho) = e^{-(\rho/\omega)^2}(\alpha\hat{x} + \beta\hat{y})$, where ρ is the radial coordinate, ω is the waist of the Gaussian mode, and $|\alpha|^2 + |\beta|^2 = 1$. By considering the overlap integral between the electric field of the emitted photon and the Gaussian mode, show that the coupling of the π-field into the cylindrically symmetric mode is uniformly zero, and the coupling of the $\sigma_{\pm}$-field is non-zero. (10 points).

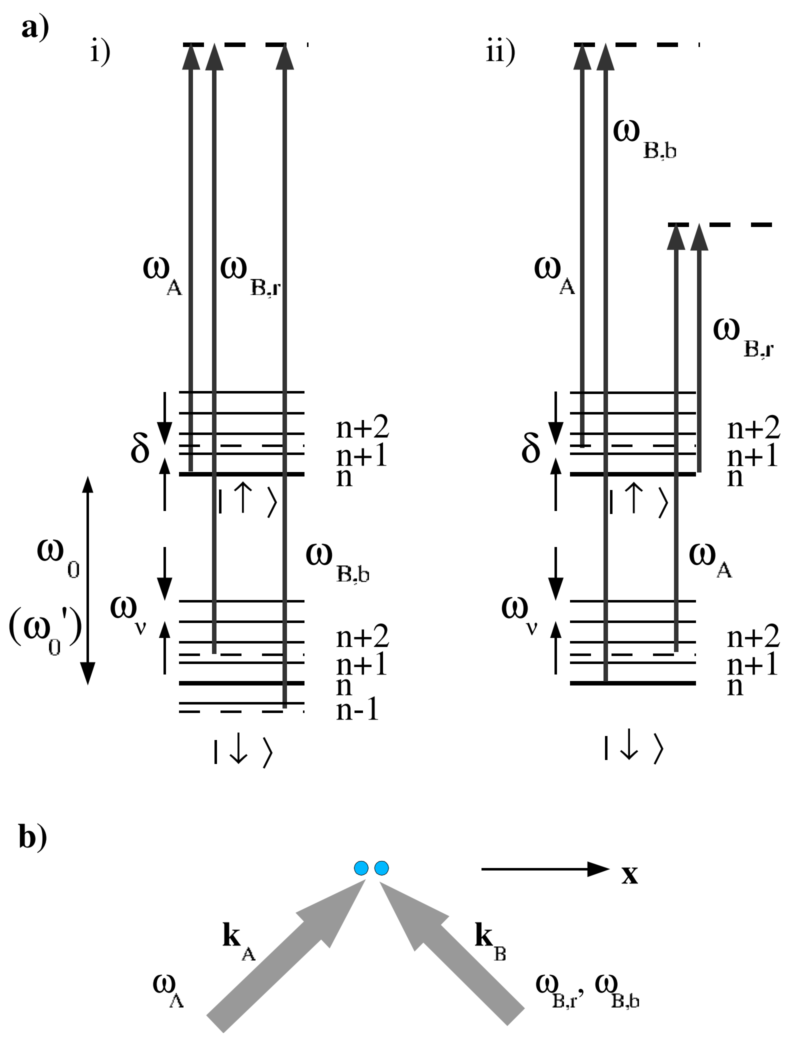
c. Next consider the case where we are collecting the photons along the equator, on x-axis, again into a single mode fiber. This time, the Gaussian mode is given by $\vec{G}(\rho) = e^{-(\rho/\omega)^2}(\alpha\hat{z} + \beta\hat{y})$. Show that the π-field coupled into the single mode fiber becomes strictly linearly polarized along the z-axis. (10 points)
[Hint: See L. J. Stephenson et al., Phys. Rev. Lett. 124, 110501 (2020).]

d. Show that the σ_+ -field coupled into the single mode fiber also becomes strictly linearly polarized, but along the y-axis. (10 points)

4. **Mølmer-Sørensen gate driven with Raman beams:** In class, we considered the case where Mølmer-Sørensen gate was driven by two fields, red- and blue-detuned from the sidebands of the qubit transition. Here, we consider the case where the qubit transitions are driven by Raman transitions. Since each of the red- and blue-detuned transitions is now driven by two fields, there are four fields involved in driving the gate. Since each Raman transition must be able to drive a sideband transition, the momentum difference of the two Raman beams must be non-zero. Here we consider each pair of the Raman beams to be arranged in a counter-propagating geometry.

    a. Re-write each of Eqs. 11.37 and 11.38 in the lecture notes with both pairs of Raman beams and their phases (total of four beams). Derive the critical parameters involved in the Mølmer-Sørensen gate operator $U_{MS}(t)$, such as $\alpha(t)$ parameter and $\hat{S}$ operator in Eq. 11.48, in terms of the phases of the four Raman beams. (10 points)

b. Since the large detuning Δ from the excited state provides us the flexibility of choosing the center frequency of the laser beams (small changes in the center frequency does not change the dynamics, as long as $\Delta >> \Omega_1, \Omega_2, \delta$), we can realize the two pairs of Raman beams with only three frequencies of lasers. We use one laser beam (Beam A) incident from the left with a single frequency ω_A, and another laser beam (Beam B) incident from the right with two frequencies $\omega_{B,r}$ and $\omega_{B,b}$, where the first subscript denotes which beam the frequency is provided, and the second subscript denotes the two frequency tones on the beam ($\omega_{B,r} < \omega_{B,b}$), which can be readily realized using an optical frequency modulator on a monotone laser beam. We consider two different options for driving the two pairs of Raman beams, as shown in the (a) and (b) panels of the figure below [Figure 9 from P. J. Lee et al., J. Optics B 7, S371 (2005), although we consider counter-propagating beam geometry that is slightly different from (c)]. Discuss the direction of the momentum kick (wavevector difference) exerted from the laser beams to the atoms for the red-sideband and blue-sideband transitions in the MS gate in these two scenarios. (10 points)



c. Write down the spin phase and motional phase for these two optical arrangements, as a function of the three phases of the laser frequencies ϕ_A, $\phi_{B,r}$ and $\phi_{B,b}$. Since the two frequencies $\omega_{B,r}$ and $\omega_{B,b}$ are created by modulating a single laser beam with an optical modulator, their relative phase relationship $\Delta\phi_B = \phi_{B,r} - \phi_{B,b}$ is determined by the phase of the modulating RF source, and can be made very stable. The phase difference between the two counter-propagating laser beams, $\Delta\phi_{AB} = \phi_A - \phi_{B,i}$ (i = r, b), is determined by the optical beam paths and is difficult to achieve interferometric stability. Discuss which phase (spin phase vs. motional phase) is more stable in both of these optical arrangements. (10 points)

d. Discuss the implication of these two scenarios in realizing a stable MS gate. Discuss its implications when considering a long quantum computation, where many MS gates are needed over an extended period of time. Make quantitative arguments assuming that in a realistic scenario, the gate detuning (ϵ, as defined in the lecture notes) is about 10kHz, the timescale for the stability of the optical beam paths is ~10ms, and the total number of MS gates needed in a "long quantum computation" is 1,000-10,000. (10 points)